THE PRINCIPLES OF

EQUATION SUB-ELEMENT THEORY

VOLUME TWO OF FOUR

SECTIONS 15 THRU 23

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SECTION 15. *RST SPREADS*

RST Spreads represent an amalgamation of *root set spacings* that apply to any *given Generalized Cubic Function (GCF)*. They accrue as the z-axis becomes displaced *vertically* with respect to such curve, now considered to be *stationary*.

Hence, they depict an assortment of relative *root set* spans which exist along such *GCF* as it becomes viewed horizontally from *different elevations*.

More specifically, *RST Spreads* may be distinguished as *deviation* from a ***three dimensional space norm*** where, for purposes of this treatise:

- Such ***norm***, or ***benchmark*** is to be represented as the *function* for *Equation 22* as follows, selected because *RST Spreads* become *useful* when they are categorized, or assembled, with respect to *3θ Cubic Functions* which they modify, or belong to:

  $$z^3 - 3\zeta z^2 - 3z + \zeta = y \qquad \textit{[Ref. Equation 22]}$$

  Where,

  $$z_R = \tan\theta_R = R\tan\theta = \tan\theta$$

  $$z_S = \tan\theta_S = S\tan\theta = \tan(\theta + 120^o)$$

  $$z_T = \tan\theta_T = T\tan\theta = \tan(\theta + 240^o)$$

  $$\theta_R + \theta_S + \theta_T = 3\theta + 360^o = 3\theta$$

- A ***three dimensional space norm*** is to be represented via the ***volumetric expletive RST***, otherwise expressed as the negative of *coefficient 'D'*, specified as follows in the *Characteristic Cubic Equation*:

  $$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

  $$AS^3 + BS^2 + CS + D = 0$$

  $$AT^3 + BT^2 + CT + D = 0$$

  Where,

  $$A = 1$$

  $$B = -(R + S + T)$$

  $$C = RS + RT + ST$$

  $$D = -RST$$

For the particular case presented above, *R equals unity*. Hence, the volume depicted by RST then reduces to a two-dimensional quantity which easily can portray any given three-space relationship. Such selection serves both as an illustrative and mathematical convenience.

However, any other arbitrary value of R may be assigned instead, depending upon what goal the mathematician sets.

15.1. Charting of *Cubic Curves* that Satisfy Certain *Spread Conditions*.

Roots for any *Cubic Function* signify *locations where* such *Curve* crosses the z-axis*.* **Of note**, relative spacing between such roots *exactly repeats itself* on associated *Cubic Functions*.

Equation Sub-element Theory enables a charting of this manifestation to occur via the method of ***RST Spreads***.

It may ***seem*** as though this already has been accomplished, however, since *Cubic Equations* may easily be ascertained from *any* ***pre-established*** *root spacings*, as follows:

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

Such that,

$$z^3 - (z_R + z_S + z_T)z^2 + (z_R z_S + z_R z_T + z_S z_T)z - z_R z_S z_T = 0$$

But, since *Cubic Curves* represent a *mapping of* sets of respective z_R, z_S *and* z_T *spacings* -- one set for each elevation, or arbitrarily chosen y value upon each curve, most importantly, this above relationship does *not* capture such *spread conditions* for $y \neq 0$; that is for the applicable *Cubic Function*. Hence, it cannot relate such quantified spreads with respect to other respective, pre-established *root structures*; nor can it account for the relative positioning of such desired spreads with respect to the origin.

For any arbitrary elevation, this *three-fold mapping* of z values does not apply within *zones* upon the *Cubic Curve* where two of the roots are imaginary, which, by the way, occurs upon every *Cubic Curve* (depending upon its placement of the z-axis). Such zones exist both above the relative high point on the cubic curve where the slope equals zero, and below its relative low point where the slope also equals zero. In these locations, only a singular root exists as the *Cubic Curve* moves out to relative positive and negative infinities, respectively.

Below, an accounting of how such determinations are accomplished via *RST Spread Philosophy* is demonstrated for a particular spacing when:

- $R = 1$
- $S = 4$
- $T = 1/2$

Since, as discussed above, R is represented as unity:

$$\theta_R + \theta_S + \theta_T = \theta + \theta_S + \theta_T = 3\theta$$
$$\theta_S + \theta_T = 2\theta$$

Such that when $\theta_S = \omega$,

$$z_S = \tan\theta_S = S\tan\theta = 4\tan\theta = \tan\omega$$

$$z_T = \tan\theta_T = T\tan\theta = \frac{1}{2}\tan\theta = \tan(2\theta - \omega)$$

$$\frac{1}{2}\tan\theta = \frac{\tan(2\theta) - \tan\omega}{1 + \tan 2\theta \tan\omega}$$

$$= \frac{\frac{2\tan\theta}{1-\tan^2\theta} - 4\tan\theta}{1 + \frac{2\tan\theta}{1-\tan^2\theta}4\tan\theta}$$

$$= \frac{2\tan\theta - (1-\tan^2\theta)4\tan\theta}{1-\tan^2\theta + (2\tan\theta)4\tan\theta}$$

$$= \frac{2\tan\theta(2\tan^2\theta - 1)}{1 + 7\tan^2\theta}$$

$$\frac{1}{2} = \frac{2(2\tan^2\theta - 1)}{1 + 7\tan^2\theta}$$

Cross multiplying yields:

$$1 + 7\tan^2\theta = 4(2\tan^2\theta - 1)$$
$$5 = \tan^2\theta$$
$$\sqrt{5} = \tan\theta$$
$$65.90515745^o = \theta$$

The governing *Family Cubic Function* is developed in *Table 24* shown below. Notice that its root structure matches the *RST Spread* allotted in the example rendered above.

Table 24. Determination of *Generalized Cubic Function* Coefficients.

ANGLES		TANGENT FUNCTIONS		COEFFICIENTS			COEFFICIENT COMBINATIONS		
θ	3θ	tan θ	$\zeta = \tan(3\theta)$	R	S	T	$R+S+T$	$RS+RT+ST$	RST
65.90515745	197.7154724	2.236067978	0.319438283	1	4	0.5	5.5	6.5	2

$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$ *GENERALIZED CUBIC FUNCTION COEFFICIENTS*			CUBIC EQUATION ROOTS		
$\beta = -(R+S+T)\tan\theta$	$\gamma = (RS+RT+ST)\tan^2\theta$	$\delta = -RST\tan^3\theta$	$z_R = R\tan\theta$	$z_S = S\tan\theta v$	$z_T = T\tan\theta$
-12.29837388	32.5	-22.36067978	2.23606798	8.944271911	1.11803399

So, the *Family Function*, or representative *Generalized Cubic Function,* is established as:

$$z^3 - 12.29837388z^2 + 32.5z - 22.36067978 = y$$

Check

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{-22.36067978 + 12.29837388}{1 - 32.5}$$

$$= \frac{-10.0623059}{-31.5}$$

$$= .319438282$$

$$3\theta = 197.7154723^o$$

$$\theta = 65.90515745^o$$

For the associated *Intermediate Cubic Function*:

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$= -\sqrt{(-12.29837388)^2 - 3(32.5)}$$

$$= -7.33143915$$

$$\nu = \frac{1}{27}[2\beta^3 + (2\beta^2 - 6\gamma)\sqrt{\beta^2 - 3\gamma} - 9\beta\gamma + 27\delta]$$

$$= \frac{1}{27}[2\beta^3 + 2(\beta^2 - 3\gamma)(-\sigma) - 9\beta\gamma + 27\delta]$$

$$= \frac{1}{27}[2\beta^3 - 2(\beta^2 - 3\gamma)\sigma - 9\beta\gamma + 27\delta]$$

$$= \frac{1}{27}[2\beta^3 - 2(\sigma^2)\sigma - 9\beta\gamma + 27\delta]$$

$$= \frac{1}{27}[2\beta^3 - 2\sigma^3 - 9\beta\gamma + 27\delta]$$

$$= \frac{1}{27}[2(-12.29837388)^3 - 2(-7.33143915)^3 - 9(-12.29837388)(32.5) + 27(-22.36067978)]$$

$$= \frac{1}{27}(-3720.258101 + 788.1297008 - +3597.27436 - 603.7383541)$$

$$= \frac{61.40760564}{27}$$

$$= 2.274355764$$

Hence the *Intermediate Cubic Function* is established as:

$$z'^3 + \sigma z'^2 + \nu = y'$$

$$z'^3 - 7.331439156z'^2 + 2.274355764 = y'$$

From above, the *Parent Cubic Function* is determined as follows:

$$z'^3 - 7.331439156z'^2 = y' - 2.274355764$$

$$z'^3 - 7.331439156z'^2 = y''$$

$$z''^3 + \sigma z''^2 = y''$$

In the above case, the *relocated origin* is displaced vertically, but not horizontally, such that $z''=z'$; whereby,

$$z'^3 + \sigma z'^2 = y''$$

Notice that the *roots* of the *Parent Cubic Function*, ascertained when setting its *function* equal to zero, may be determined via reduced *Linear Equation* as follows:

Where,

$$z'^3 + \sigma z'^2 = 0$$

$$z' + \sigma = 0$$

$$z' = -\sigma$$

Lastly, the *norm* is established as the *function* for *Equation 22*, as follows:

$$z^3 - 3\zeta z^2 - 3z + \zeta = y \qquad \textit{[Ref. Equation 22]}$$

$$z^3 + \beta z^2 + \gamma z + \delta = y$$

Where,

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$\sigma^2 = \beta'^2 - 3\gamma$$

Or,

$$\beta = \sqrt{\sigma^2 + 3\gamma}$$

$$= 6.689544081$$

$$\delta = -\frac{\beta}{3} = -2.229848027$$

Accordingly, the associated *3θ Cubic Function* is:

$$z^3 + 6.689544081z^2 - 3z - 2.229848027 = y$$

All curves are *virtually identical*, except for the fact that they are simply *translated*, or moved to different locations about the origin. This is evidenced by examining Δ and ε sets *(Ref. Section 14.2.2)* for each respective curve and verifying that they are all *exactly the same*, including the Δ and ε set result for the associated *3θ Cubic Function* case, as computed in *Table 25* below:

Table 25. First Associated Curve Properties Chart.

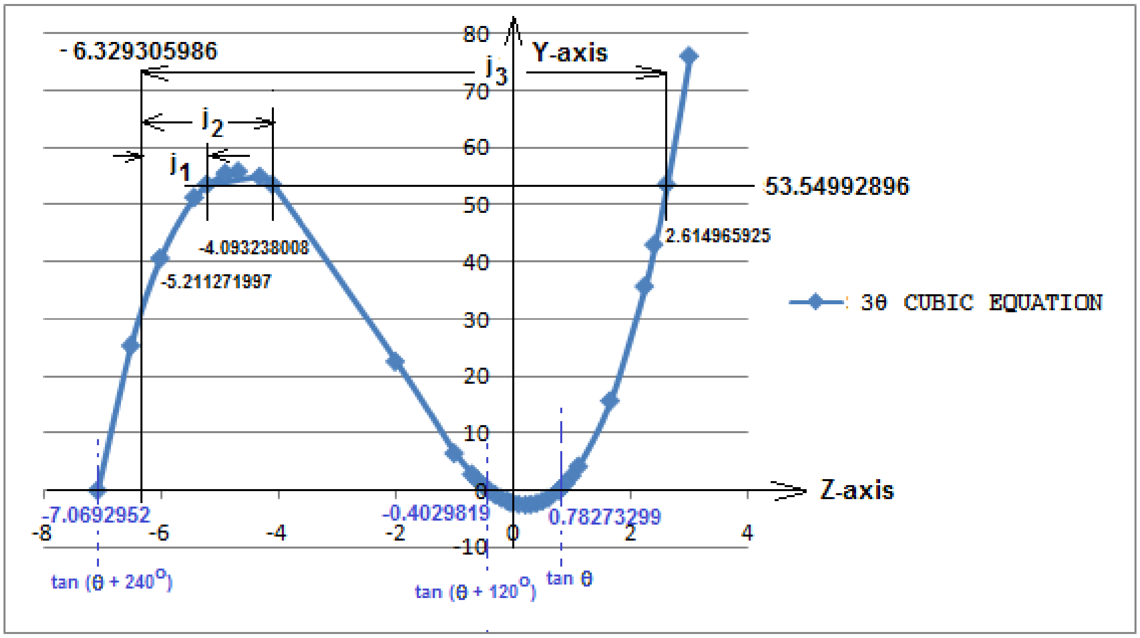
	$z^3+\beta' z^2+\gamma' z+\delta' = y_{TRANSFORMED}$	$z'^3+\sigma z'^2+\nu = y'$	$z'^3+\sigma z'^2 = y''$	$z^3-3\zeta' z^2-3z+\zeta' = y$
	FAMILY CURVE	**ASSOCIATED INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
	$z^3-12.29837388z^2+32.5z-22.36067978 = y$	$z'^3-7.33143915z'^2+2.274355764 = y'$	$z'^3-7.33143915z'^2 = y''$	$z^3+6.689544081z_2-3z-2.229848027 = y$
$\Delta=\frac{2}{3}\sqrt{\beta'^2-3\gamma'}=-\frac{2}{3}\sigma$ *[Ref. Section 14.2.2]*	4.887626104	4.8876261	4.8876261	4.8876261
$\varepsilon=\frac{4}{27}(\sqrt{\beta'^2-3\gamma'})^3=\frac{4}{27}(-\sigma)^3$ *[Ref. Section 14.2.2]*	58.37997856	58.37997843	58.37997843	58.37997843
$z_A=\frac{1}{3}[-\beta'+\sqrt{\beta'^2-3\gamma'}]$ *[Ref. Section 14.2.1]*	6.543271009	*Not Applicable*	*Not Applicable*	0.213965023
$z_B=\frac{1}{3}[-\beta'-\sqrt{\beta'^2-3\gamma'}]$ *[Ref. Section 14.2.1]*	1.655644909	*Not Applicable*	*Not Applicable*	-4.673661077

Table 25 entries rendered above are to be interpreted as follows:

- *Second column results* reflect a β′ value of −12.29837388 and γ′ value of +32.5 applied to first column equations
- *Third and fourth column results* reflect a σ value of −7.33143915 applied to first column equations
- *Fifth column results* reflect a β′ value of +6.689544081 and γ′ value of −3 applied to first column equations

The associated *3θ Cubic Function* is portrayed below in *Figure 23*.

Figure 23. First Associated 3θ Cubic Function Figure.



For this curve, $\zeta' = \tan 3\theta = -\beta/3$.

So,

$$\tan 3\theta = -6.689544081/3$$
$$= -2.229848024$$
$$3\theta = 114.1543767°$$
$$\theta = 38.0514589°$$
$$\tan \theta = 0.782732993$$
$$\tan (\theta + 120°) = -0.40298187$$
$$\tan (\theta + 240°) = -7.0692952$$

In *Figure 23*, these roots are displayed along the abscissa. Moreover, *Figure 23* discloses that three-*fold* values for the *3θ Cubic Curve* for the particular condition when 'y' equals 53.54992896 (as determined below) are as follows:

- $z_S = +2.614965925$
- $z_R = -4.093238008$
- $z_T = -5.211271997$

Notice that j_1, j_2, and j_3 distances to this above given root set are marked off from a z value of -6.329305986 where it just so happens to turn out that:

$$j_1 = -5.211271997 - (-6.329305986) = 1.118033989$$
$$j_2 = -4.093238008 - (-6.329305986) = 2.236067978$$

j_3 = +2.614965925 – (-6.329305986) = 8.944271911

This spacing equates to the pre-established spacing given above via the following ratios:

- j_1/j_2 = 1.118033989/2.236067978= ½ to 1 = T
- j_2/j_2 = 2.236067978/2.236067978= 1 to 1 = R
- j_3/j_2 = 8.944271911/2.236067978= 4 to 1 = S

Lastly, the value of -6.329305986 adopted in the above analysis is determined as the *horizontal offset* calculated from the origin to z_B for the *3θ Cubic Curve* minus the *horizontal offset* from the origin to z_B for the associated *Family Curve*, where:

$$[z_B]_{3\theta\text{Cubic Curve}} = \frac{1}{3}[-\beta' - \sqrt{\beta'^2 - 3\gamma'}]$$

$$= \frac{1}{3}[-6.689544081 - \sqrt{(6.689544081)^2 - 3(-3)}]$$

$$= -4.673661077$$ (*ref. Table 25*)

$$[z_B]_{\text{Family Curve}} = \frac{1}{3}[-\beta' - \sqrt{\beta'^2 - 3\gamma'}]$$

$$= \frac{1}{3}[12.29837388 - \sqrt{(-12.29837388)^2 - 3(+32.5)}]$$

$$= +1.655644909$$ (*ref. Table 25*)

$$[z_B]_{3\theta\text{CubicCurve}} - [z_B]_{\text{Family Curve}} = -4.673661077 - 1.655644908$$

$$= -6.329305986$$

It becomes easier to comprehend just how the above mentioned relationships may be ascertained when considering the *3θ Cubic Function* in relation to its other associated curves. As introduced above, such curves are *virtually identical* with the exception that they are translated in an organized fashion about a *fixed origin*. The amount of translation is simply accounted for by viewing either of the relative displacements of the readily identifiable two points of zero slope located upon each displaced curve via the equation:

$$z_{A,B} = \frac{1}{3}[-\beta' \pm \sqrt{\beta'^2 - 3\gamma'}]$$

In the illustrative example presented above, the analysis moved from a determination of the *Family Cubic Function*, to its *associated Intermediate and Parent Cubic Functions*, and lastly, to its related *3θ Cubic* norm*.* This is charted in *Table 26 below*:

Table 26. First Determination for *3θ Cubic Function aside its Associated Curves.*

	GENERALIZED CUBIC FUNCTION			
	$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	INTERMEDIATE CURVE	PARENT CURVE	3θ CUBIC CURVE
z	$z^3 - 12.29837388z^2 + 32.5z - 22.36067978 = y$	$z'^3 - 7.33143915z'^2 + 2.274355764 = y'$	$z'^3 - 7.33143915z'^2 = y''$	$z^3 + 6.689544081 z_2 - 3z - 2.229848027 = y$
10	72.80193222	269.1304411	266.856085	1636.72456
9	2.97103594	137.427785	135.1534289	1241.623223
8.944271911	0	131.300977	129.0266209	1221.642616
8	-37.4566081	45.06225054	42.7878944	913.9009732
7.288627002	-51.61851405	0	-2.274355764	718.4817997
6.543271009	-56.10562243	-31.47058636	-33.7449425	544.6953178
5	-42.32002678	-56.01162261	-58.28597875	275.008754
4	-25.13466186	-51.02867026	-53.3030264	156.8028573
3.5	-16.39075981	-44.66077344	-46.93512959	112.092067
3.259573087	-12.46015616	-40.98847445	-43.26283059	93.69897907
3	-8.5460447	-36.70859621	-38.98295235	75.9760487
2.926239754	-7.510339186	-35.44683369	-37.72118983	71.33022712
2.614965925	-3.589883887	-29.97710844	-32.25146459	53.54992896
2.236067978	0	-23.20249972	-25.47685587	35.69000834
1.725708115	2.238711101	-14.41989486	-52.32473534	17.65422509
1.655644909	2.274356128	-13.28390437	-15.55826051	15.67871604
1.283085371	1.20503642	-7.683097488	-9.957453632	7.04630006
1.118033989	0	-5.492400309	-7.766756453	4.175522596

	GENERALIZED CUBIC FUNCTION			
	$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	**INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
z	$z^3 - 12.29837388z^2 + 32.5z - 22.36067978 = y$	$z'^3 - 7.33143915z'^2 + 2.274355764 = y'$	$z'^3 - 7.33143915z'^2 = y''$	$z^3 + 6.689544081z_2 - 3z - 2.229848027 = y$
1	-1.15905366	-4.057083006	-6.33143915	2.459696054
0.9	-2.343362623	-2.935109568	-5.209465712	1.217682679
0.8	-3.719639063	-1.905764912	-4.180121056	0.163460185
0.782732993	-3.977156015	-1.737845802	-4.012201946	0
0.740426913	-4.633241651	-1.339047762	-3.613403906	-0.377780811
0.6	-7.072094377	-0.14896195	-2.423318094	-1.405612158
0.580423069	-7.444601498	0	-2.274355764	-1.521931177
0.5	-9.06027325	0.566496357	-1.707859788	-1.932462007
0.4	-11.2644196	1.16532588	-1.109030264	-2.295520974
0.3	-13.69053343	1.641526621	-0.632829524	-2.50078906
0.259573087	-14.73570702	1.797866625	-0.476489519	-2.540348368
0.213965023	-15.96005323	1.94851084	-0.325845304	-2.555693331
0.1	-19.23266352	2.202041753	-0.072314392	-2.461952586
0	-22.36067978	2.274356144	0	-2.229848027
-0.05	-24.01655071	2.255902546	-0.018453598	-2.063249167
-0.1	-25.73466352	2.200041753	-0.074314392	-1.863952586
-0.139327752	-27.13027515	2.129331955	-0.145024189	-1.684710514
-0.2	-29.36061474	1.973098578	-0.301257566	-1.370266264
-0.240426913	-30.89936108	1.836663667	-0.437692477	-1.135775424
-0.3	-33.24453343	1.587526621	-0.686829524	-0.75478906
-0.402981869	-37.5202194	1.018329586	-1.256026558	0

	GENERALIZED CUBIC FUNCTION			
	$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	**INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
z	$z^3 - 12.29837388z^2 + 32.5z - 22.36067978 = y$	$z'^3 - 7.33143915z'^2 + 2.274355764 = y'$	$z'^3 - 7.33143915z'^2 = y''$	$z^3 + 6.689544081 z_2 - 3z - 2.229848027 = y$
-0.53761092	-43.54296163	0	-2.274355764	1.161050301
-0.55	-44.12231288	-0.109779199	-2.384135343	1.277364058
-0.6	-46.50409438	-0.58096195	-2.855318094	1.762387842
-0.7	-51.47988298	-1.66104904	-3.935405184	2.805028573
-1	-68.15905366	-6.057083006	-8.33143915	6.459696054
-1.274291885	-85.81472957	-11.69980038	-5.334120342	10.38642141
-1.774291885	-124.3274897	-26.39150221	-1.171884668	18.56679054
-2	-144.5541753	-35.05140046	-37.3257566	22.5283283
-2.466633274	-192.3606946	-57.33985709	-59.61421323	30.86342233
-3	-257.5460447	-90.70859621	-92.98295235	39.9760487
-4.093238008	-430.0257729	-189.14151	-191.4158661	53.54992895
-4.673661077	-544.976655	-259.954343	-262.2286992	55.8242851
-4.8876261	-591.7629721	-289.625536	-58.37997843	55.47884872
-5	-617.3200268	-306.0116226	-308.2859788	55.008754
-5.211271997	-667.2427032	-338.3525135	-340.6268697	53.54992896
-6.5	-1027.841976	-582.1039479	-584.3783041	25.2783894
-7.069295204	-1220.010771	-717.4014026	-719.6757587	0
-8	-1581.456608	-978.9377495	-981.2121056	-62.09902684
-9	-2040.028964	-1320.572215	-1322.846571	-162.3767775
-10	-2577.198068	-1730.869559	-1733.143915	-303.2754399

First, notice that the roots afforded for the *Family Curve* in *Table 26* match those determined in *Table 24*. They are easily identifiable because they indicate respective y values of zero.

For the *Associated Intermediate Curve*, three-fold real roots exist at the following 'z' values:

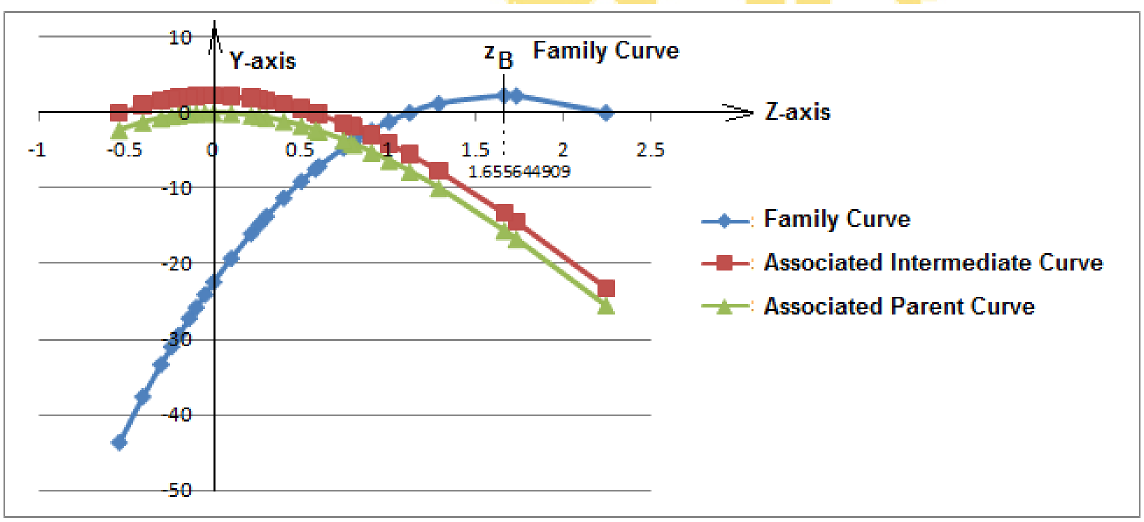
- 7.288627002
- 0.580423069
- -0.53761092

Such roots are determined by subtracting the value z_B for the *Family Curve* given in *Table 25* from the respective root values specified in *Table 26* as follows:

- 8.944271911 - 1.655644909 = 7.288627002
- 2.23606798 - 1.655644909 = 0.580423069
- 1.11803399 - 1.655644909 = -0.53761092

Now the *Associated Intermediate Curve* is displaced horizontally to the left of the *Family Curve* by a distance of z_B (*ref. Figure 24*). This is because z_B for the *Associated Intermediate Curve* aligns with the y-axis. Likewise, since the *Associated Intermediate Curve* is an exact copy of the *Family Curve*, its roots too must be located by the same amount to the left of the respective roots for the *Family Curve*. Hence, the subtraction by an amount z_B = 1.655644909 (*ref. Table 25*) is applied to the roots of the *Family Curve* in order to determine the roots for the *Associated Intermediate Curve*.

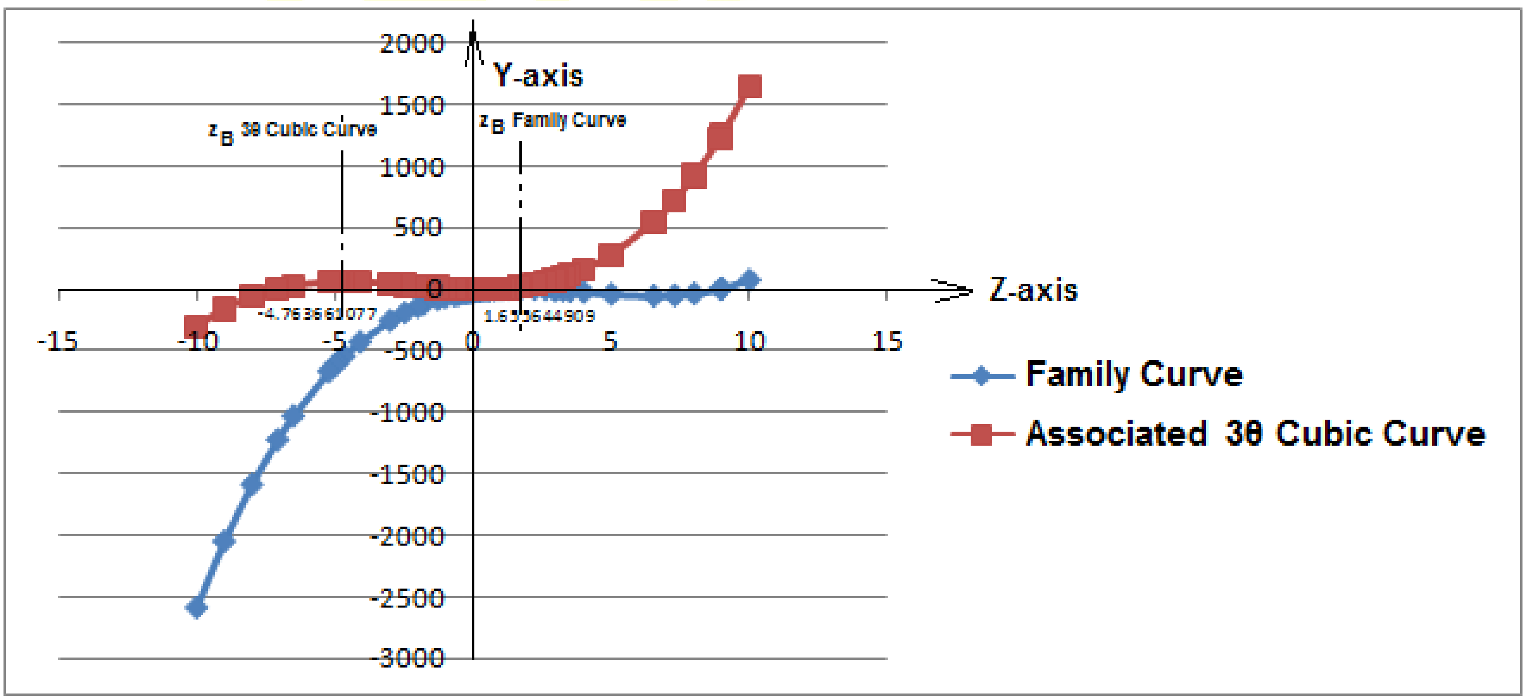
Figure 24. First Relative Positioning of *Family Curve* to *Other Associated Curves*.



For the *three root values* specified above, respective 'y' values for the *Associated Intermediate Curve* all equal zero *(ref. Table 26)*. Accordingly, respective 'z' values for the *Associated Parent Curve* all must occur at $y = -2.274355764$ *(Ref. Table 26)* since this curve is an exact copy of the *Associated Intermediate Curve*, with its only exception being that it *rides* a distance a ν below it.

Horizontal spacing between the *Family Curve* and its *Associated 3θ Cubic Curve* again is related in terms of separation between their relative z_B values *(Ref. Figure 25)*. Since the two curves are identical, each independent set of z_R, z_S, and z_T values for respective elevations on the *Family Curve* is exhibited in exactly the same fashion on the *3θ Cubic Curve*, with the only exception being that on the latter curve, they are displaced horizontally to the left an amount equal to their relative z_B separation. For this analysis, vertical displacement is irrelevant.

Figure 25. First Relative Positioning of *Family Curve* to *Associated 3θ Cubic Curve*.



The three roots of the *Family Curve*, repeat themselves upon the *Associated 3θ Cubic Curve*, but at the following z-axis locations *(ref. Table 25, Table 26 and Figure 23)*:

$$
\begin{aligned}
z_{S'} &= 8.944271911 - [(z_B)_{\text{Family Curve}} - (z_B)_{\text{Associated } 3\theta \text{ Cubic Curve}}] \\
&= 8.944271911 - [1.655644909 - (-4.673661077)] \\
&= 8.944271911 - 6.329305986 \\
&= +2.614965925
\end{aligned}
$$

$$z_{R'} = 2.236067978 - [(z_B)_{\text{Family Curve}} - (z_B)_{\text{Associated } 3\theta \text{ Cubic Curve}}]$$
$$= 2.236067978 - [1.655644909 - (-4.673661077)]$$
$$= 2.236067978 - 6.329305986$$
$$= -4.093238008$$

$$z_{T'} = 1.118033989 - [(z_B)_{\text{Family Curve}} - (z_B)_{\text{Associated } 3\theta \text{ Cubic Curve}}]$$
$$= 1.118033989 - [1.655644909 - (-4.673661077)]$$
$$= 1.118033989 - 6.329305986$$
$$= -5.211271997$$

Each of these above three values lies horizontally with respect to one another, validated as follows:

$$z^3+6.689544081z^2-3z-2.229848027=y$$

$$z_{S'}^3+6.689544081z_{S'}^2-3z_{S'}-2.229848027=y$$
$$(2.614965925)^3 + 6.689544081(2.614965925)^2 - 3(2.614965925) - 2.229848027 = y$$
$$53.54992896 = y$$

$$z_{R'}^3+6.689544081z_{R'}^2-3z_{R'}-2.229848027=y$$
$$(-4.093238008)^3 + 6.689544081(-4.093238008)^2 - 3(-4.093238008) - 2.229848027 = y$$
$$53.54992896 = y$$

$$z_{T'}^3+6.689544081z_{T'}^2-3z_{T'}-2.229848027=y$$
$$(-5.211271997)^3 + 6.689544081(-5.211271997)^2 - 3(-5.211271997) - 2.229848027 = y$$
$$53.54992896 = y$$

Equation 36 is applied below in order to check that for the *Associated 3θ Cubic Function* $z^3 + 6.689544081z^2 - 3z - 2.229848027 = y$ tan 3θ indeed is denoted by its last term:

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta = \frac{-2.229848027 - 6.689544081}{1-(-3)} = -2.229848027$$

Where,

$$\zeta = -\beta/3$$
$$= -6.689544081/3$$
$$= -2.229848027$$
$$3\theta = 114.1543767^\circ$$
$$\theta = 38.0514589^\circ$$

Then, its three *roots*, as portrayed in *Table 26*, are:

$$\tan\theta = 0.782732993$$

$$\tan(\theta + 120^\circ) = -0.402981869$$

$$\tan(\theta + 240^\circ) = -7.069295204$$

In addition to the set of curves established above, a **second set** of *completely different Cubic Curves* exists that characterizes the same RST spread of ½ to 1 to 4. This *second set of Curves* is determined when executing the following aforementioned relationship:

$$(z-R)(z-S)(z-T) = 0$$

$$z^3 - (R+S+T)z^2 + (RS+RT+ST)z - RST = 0$$

Since, R represents a specified root, this above result is synonymous with the *Characteristic Cubic Equation* rendered below:

$$AR^3 + BR^2 + CR + D = 0$$ *[Ref. Equation 31]*

Hence its coefficients are:

$$z^3 - (R+S+T)z^2 + (RS+RT+ST)z - RST = 0$$

$$z^3 - (1+4+0.5)z^2 + [4+0.5+4(0.5)]z - 4(0.5) = 0$$

$$z^3 - 5.5z^2 + 6.5z - 2 = 0$$

Or,

$$R^3 - 5.5R^2 + 6.5R - 2 = 0$$

$$S^3 - 5.5S^2 + 6.5S - 2 = 0$$

$$T^3 - 5.5T^2 + 6.5T - 2 = 0$$

The fact that this curve set is independent, or different from the curve set addressed above, is verified by comparing their respective Δ and ε lengths. This is accomplished by calculating such lengths for the *second Curve* developed directly above as:

$$\Delta = \frac{2}{3}\sqrt{\beta'^2 - 3\gamma'}$$

$$= \frac{2}{3}\sqrt{(-5.5)^2 - 3(6.5)}$$

$$= 2.185812841$$

So,

$$\frac{3}{2}\Delta = \sqrt{\beta'^2 - 3\gamma'}$$

$$\frac{9}{4}\Delta^2 = \beta'^2 - 3\gamma'$$

Where,

$$\varepsilon = \frac{4}{27}(\sqrt{\beta'^2 - 3\gamma'})^3$$

$$= \frac{4}{27}(\frac{3}{2}\Delta)^3$$

$$= -\frac{4}{27}(\frac{27}{8})\Delta^3$$

$$= \frac{1}{2}\Delta^3$$

$$= \frac{1}{2}(2.185812841)^3$$

$$= +5.22166401$$

$$m = -\frac{\varepsilon}{\Delta}$$

$$= -\frac{2}{9}(\beta'^2 - 3\gamma')$$

$$= -\frac{2}{9}(\frac{9}{4}\Delta^2)$$

$$= -\frac{\Delta^2}{2}$$

$$= -\frac{(2.185812841)^2}{2}$$

$$= -2.3888888889$$

Compared against the values rendered in *Table 25*, where Δ equals 4.887626104 and ε equals 58.37997856, such that m equals $-\Delta^2/2$, or $-(-4.887626104)^2/2 = -11.9444447$, it is shown that these curves are entirely different. However, the ratio of the slopes between the *two sets* turns out to be exactly 5, meaning that the first set of *Curves* is five times as steep as the *second set*.

This is verified by viewing the ratio of Δ values between the *two Curves* as follows:

$$\frac{\Delta_{Associated\,3\theta\,Cubic\,Function}}{\Delta_{Characteristic\,Cubic\,Equation}} = \frac{4.8876261}{2.185812841}$$

$$= 2.236067978$$

$$= \sqrt{5}$$

Then,

$$\frac{m_{Associated\,3\theta CubicFunction}}{m_{CharacteristicCubicEquation}} = \frac{\frac{(\Delta_{Associated\,3\theta CubicFunction})^2}{2}}{\frac{(\Delta_{CharacteristicCubicEquation})^2}{2}}$$

$$= (\frac{\Delta_{Associated\,3\theta CubicFunction}}{\Delta_{CharacteristicCubicEquation}})^2$$

$$= (\sqrt{5})^2$$

$$= 5$$

Below, the same type of comprehensive assessment is rendered for this *second set of Curves* – namely, the *Characteristic Cubic Function Set* that was applied to the first *Generalized Cubic Function Set of Curves*.

First, the associated *Intermediate Cubic Function* is determined as follows:

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$= -\sqrt{(-5.5)^2 - 3(+6.5)}$$

= -3.278719262

$$\nu = \frac{1}{27}[2\beta^3 - 2\sigma^3 - 9\beta\gamma + 27\delta]$$

$$= \frac{1}{27}[2(-5.5)^3 - 2(-3.278719262)^3 - 9(-5.5)(+6.5) + 27(-2)]$$

$$= \frac{1}{27}[-332.75 + 70.49246414 + 321.75 - 54]$$

$$= \frac{1}{27}[5.492464136]$$

= +0.203424598

Hence the *Intermediate Cubic Function* and *Parent Cubic Function* are established respectively as:

- $z^3 - 3.278719262z^2 + +0.203424598 = y$
- $z^3 - 3.278719262z^2 = y$

Lastly, the *norm* is established as the *function* for *Equation 22*, as follows:

$$z^3 - 3\zeta z^2 - 3z + \zeta = y$$ *[Ref. Equation 22]*

$$z^3 + \beta z^2 + \gamma z + \delta = y$$

Where,

$$\beta = \sqrt{\sigma^2 + 3\gamma'}$$

$$= \sqrt{(-3.278719262)^2 + 3(-3)}$$

= 1.322875656

$$\delta = -\frac{\beta}{3} = -0.440958552$$

Accordingly, the associated *3θ Cubic Function* is:

- $z^3 + 1.322875656z^2 - 3z - 0.440958552 = y$

All curves are *virtually identical*, except for the fact that they are simply *translated*, or moved to different locations about the origin. This is evidenced by examining Δ and ε sets for each respective curve and verifying that they are all *exactly the same*, including the Δ and ε set result for the associated *3θ Cubic Function* case, as computed in *Table 27* below:

Table 27. Second Associated Curve Properties Chart.

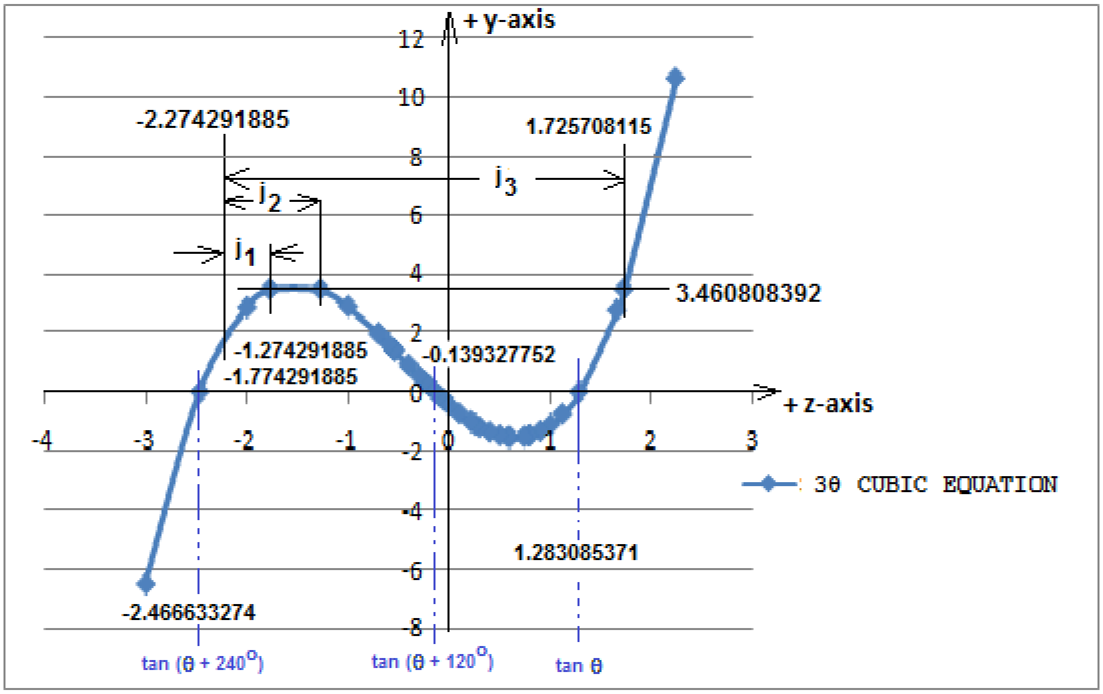
	$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	**INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
	$z^3 - 5.5z^2 + 6.5z - 2 = y$	$z'^3 - 3.278719262z'^2 + 0.203424598 = y'$	$z'^3 - 3.278719262z'^2 = y''$	$z^3 + 1.322875656z^2 - 3z - 0.440958552 = y$
$\Delta = \frac{2}{3}\sqrt{\beta'^2 - 3\gamma'} = -\frac{2}{3}\sigma$ *[Ref. Section 14.2.2]*	2.185812841	2.185812841	2.185812841	2.185812841
$\varepsilon = \frac{4}{27}(\sqrt{\beta'^2 - 3\gamma'})^3 = \frac{4}{27}(-\sigma)^3$ *[Ref. Section 14.2.2]*	5.22166401	5.22166401	5.22166401	5.22166401
$z_A = \frac{1}{3}[-\beta' + \sqrt{\beta'^2 - 3\gamma'}]$ *[Ref. Section 14.2.1]*	2.926239754	*Not Applicable*	*Not Applicable*	0.651947869
$z_B = \frac{1}{3}[-\beta' - \sqrt{\beta'^2 - 3\gamma'}]$ *[Ref. Section 14.2.1]*	0.740426913	*Not Applicable*	*Not Applicable*	-1.533864973

Table 27 entries rendered above are to be interpreted as follows:

- *Second column results* reflect a β′ value of −5.5 and γ′ value of +6.5 applied to first column equations
- *Third and fourth column results* reflect a σ value of −3.278719262 applied to first column equations
- *Fifth column results* reflect a β′ value of +1.322875655 and γ′ value of −3 applied to first column equations

The associated *3θ Cubic Function* is portrayed below in *Figure 26*.

Figure 26. Second Associated 3θ Cubic Function Figure.



For this curve, $\zeta' = \tan 3\theta = -\beta/3$.

So,

$$\tan 3\theta = -1.322875655/3$$
$$= -0.440958552$$

$$3\theta = 156.2045089^\circ$$
$$\theta = 52.06816963^\circ$$

$$\tan\theta = 1.283085371$$
$$\tan(\theta + 120^\circ) = -0.139327752$$
$$\tan(\theta + 240^\circ) = -2.466633274$$

In *Figure 26*, these roots are displayed upon vertical projections which emanate from roots along the abscissa. Moreover, *Figure 26* discloses that three-*fold* values for the *3θ Cubic Curve* during the particular condition when 'y' equals 3.460808394 (as determined below) are as follows:

- $z_S = +1.725708115$
- $z_R = -1.274291885$
- $z_T = -1.774291885$

Notice that j_1, j_2, and j_3 distances to this above given root set are marked off from a z value of −2.274291885 such that:

$j_1 = -1.774291885 - (-2.274291885) = 0.5$
$j_2 = -1.274291885 - (-2.274291885) = 1$
$j_3 = +1.725708115 - (-2.274291885) = 4$

This spacing equates to the pre-established spacing given above via the following ratios:

- j_1/j_2 = ½ to 1 = T
- j_2/j_2 = 1 to 1 = R
- j_3/j_2 = 4 to 1 = S

Lastly, the value of -2.274291885 adopted in the above analysis is determined as the *horizontal offset* calculated from the origin to z_B for the *3θ Cubic Curve* minus the *horizontal offset* from the origin to z_B for the *associated Family Curve.*, where:

$$[z_B]_{3\theta\text{Cubic Curve}} = \frac{1}{3}[-\beta' - \sqrt{\beta'^2 - 3\gamma'}]$$

$$= \frac{1}{3}[-1.322875655 - \sqrt{(1.322875655)^2 - 3(-3)}]$$

$$= -1.533864973 \qquad (\textit{ref. Table 27})$$

$$[z_B]_{\text{Family Curve}} = \frac{1}{3}[-\beta' - \sqrt{\beta'^2 - 3\gamma'}]$$

$$= \frac{1}{3}[+5.5 - \sqrt{(-5.5)^2 - 3(+6.5)}]$$

$$= +0.740426913 \qquad (\textit{ref. Table 27})$$

$$[z_B]_{3\theta\text{CubicCurve}} - [z_B]_{\text{Family Curve}} = -1.533864973 - 0.740426913$$

$$= -2.274291885$$

It becomes easier to comprehend just how the above mentioned relationships may be ascertained when considering the *3θ Cubic Function* in relation to its other *associated Curves*. As introduced above, such curves are *virtually identical* with the exception that they are translated in an organized fashion about a *fixed origin*. The amount of translation is simply accounted for by viewing either of the relative displacements of the readily identifiable two points of zero slope located upon each displaced curve via the equation:

$$z_{A,B} = \frac{1}{3}[-\beta' \pm \sqrt{\beta'^2 - 3\gamma'}]$$

In the illustrative example presented above, the analysis moves from a determination of the *Family Cubic Function*, to its *associated Intermediate and Parent Cubic Functions*, and lastly, to its related *3θ Cubic* norm. This is charted in *Table* below:

Table 28. Second Determination for 3θ Cubic Function aside its Associated Curves.

	CHARACTERISTIC CUBIC FUNCTION			
	$AR^3 + BR^2 + CR + D = y$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	**INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
z	$z^3 - 5.5z^2 + 6.5z - 2 = y$	$z'^3 - 3.278719262z'^2 + 0.203424598 = y'$	$z'^3 - 3.278719262z'^2 = y''$	$z^3 + 1.322875656z^2 - 3z - 0.440958552 = y$
10	513	672.3314984	672.1280738	1101.846607
9	340	463.6271644	463.4237398	808.7119696
8.944271911	331.6795203	453.4476366	453.244212	794.0980312
8	210	302.3653918	302.1619672	572.2230834
7.288627002	140.3952458	213.2260985	213.0226739	435.1713473
6.543271009	85.19827916	139.9732342	139.7698096	316.7135428
5	18	43.23544305	43.03201845	142.6309328
4	0	11.74391641	11.54049181	72.72505194
3.5	-3.75	2.914113639	2.710689041	48.13926823
3.259573087	-4.616900235	0	-0.203424596	38.46800018
3	-5	-2.30504876	-2.508473358	29.46492235
2.926239754	-5.018239412	-2.814814813	-3.018239411	27.16498372
2.614965925	-4.73071948	-4.335351778	-4.538776376	18.64128865
2.236067978	-3.785218259	-5.009831824	-5.213256422	10.64555569
1.725708115	-2.022997018	-4.421548976	-4.624973574	3.460808394
1.655644909	-1.776300742	-4.245682001	-4.449106599	2.756708347
1.283085371	-0.60228567	-3.08200358	-3.285428178	0
1.118033989	-0.210236587	-2.497431994	-2.700856592	-0.743923462
1	0	-2.075294664	-2.278719262	-1.118082896

	CHARACTERISTIC CUBIC FUNCTION			
	$AR^3 + BR^2 + CR + D = y$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	**INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
z	$z^3 - 5.5z^2 + 6.5z - 2 = y$	$z'^3 - 3.278719262z'^2 + 0.203424598 = y'$	$z'^3 - 3.278719262z'^2 = y''$	$z^3 + 1.322875656z^2 - 3z - 0.440958552 = y$
0.9	0.124	-1.723338004	-1.926762602	-1.340429271
0.8	0.192	-1.38295573	-1.586380328	-1.482318132
0.782732993	0.197632051	-1.325793651	-1.529218249	-1.499112304
0.740426913	0.203424598	-1.188148526	-1.391573124	-1.531070769
0.6	0.136	-0.760914336	-0.964338934	-1.548723316
0.580423069	0.115389056	-0.705606939	-0.909031537	-1.541023664
0.5	0	-0.491255218	-0.694679816	-1.485239638
0.4	-0.216	-0.257170484	-0.460595082	-1.365298447
0.3	-0.518	-0.064660136	-0.268084734	-1.194899743
0.259573087	-0.6658654	0	-0.203424598	-1.113055286
0.213965023	-0.851227482	0.063116989	-0.140307609	-1.01249547
0.1	-1.404	0.171637405	-0.031787193	-0.726729795
0	-2	0.203424598	0	-0.440958552
-0.05	-2.338875	0.1951028	-0.008321798	-0.287776363
-0.1	-2.706	0.169637405	-0.033787193	-0.128729795
-0.139327752	-3.015102273	0.137072709	-0.066351889	0
-0.2	-3.528	0.064275828	-0.13914877	0.203956474
-0.240426913	-3.894600886	0	-0.203424598	0.342893244
-0.3	-4.472	-0.118660136	-0.322084734	0.551100257
-0.402981869	-5.577993269	-0.394462999	-0.597887597	0.917372642

	CHARACTERISTIC CUBIC FUNCTION			
	$AR^3 + BR^2 + CR + D = y$	$z'^3 + \sigma z'^2 + \nu = y'$	$z'^3 + \sigma z'^2 = y''$	$z^3 - 3\zeta' z^2 - 3z + \zeta' = y$
	FAMILY CURVE	**INTERMEDIATE CURVE**	**PARENT CURVE**	**3θ CUBIC CURVE**
z	$z^3 - 5.5z^2 + 6.5z - 2 = y$	$z'^3 - 3.278719262z'^2 + 0.203424598 = y'$	$z'^3 - 3.278719262z'^2 = y''$	$z^3 + 1.322875656z^2 - 3z - 0.440958552 = y$
-0.53761092	-7.239494506	-0.899592147	-1.103016745	1.398835743
-0.55	-7.405125	-0.954762979	-1.158187577	1.442836334
-0.6	-8.096	-1.192914336	-1.396338934	1.619276684
-0.7	-9.588	-1.74614784	-1.949572438	1.964250519
-1	-15	-4.075294664	-4.278719262	2.881917104
-1.274291885	-21.28312661	-7.189845092	-7.39326969	3.460808394
-1.774291885	-36.4331806	-15.70401888	-15.90744348	3.460808394
-2	-45	-20.91145245	-21.11487705	2.850544072
-2.466633274	-66.50434145	-34.75290725	-34.95633185	0
-3	-98	-56.30504876	-56.50847336	-6.535077648
-4.093238008	-189.3368876	-123.3107515	-123.5141761	-34.57755038
-4.673661077	-254.6031733	-173.5012769	-173.7047015	-59.6115427
4.8876261	15.14063759	38.63842129	38.43499669	133.2581496
-5	-297	-206.764557	-206.9679816	-77.36906715
-5.211271997	-326.763093	-230.362289	-230.5657136	-90.4057056
-6.5	-551.25	-412.9474642	-413.1508888	-199.6744621
-7.069295204	-676.1001256	-516.9379224	-517.141347	-266.4100145
-8	-918	-721.6346082	-721.8380328	-403.7769166
-9	-1235	-994.3728356	-994.5762602	-595.2880304
-10	-1617	-1327.668502	-1327.871926	-838.153393

First, notice that the *roots* afforded for the *Family Curve* in *Table 28* match those respective R = 1, S = 4, and T = ½ values enumerated above. They are easily identifiable because they indicate respective y values of zero.

For the *Associated Intermediate Curve*, three-fold real roots exist at the following 'z' values:

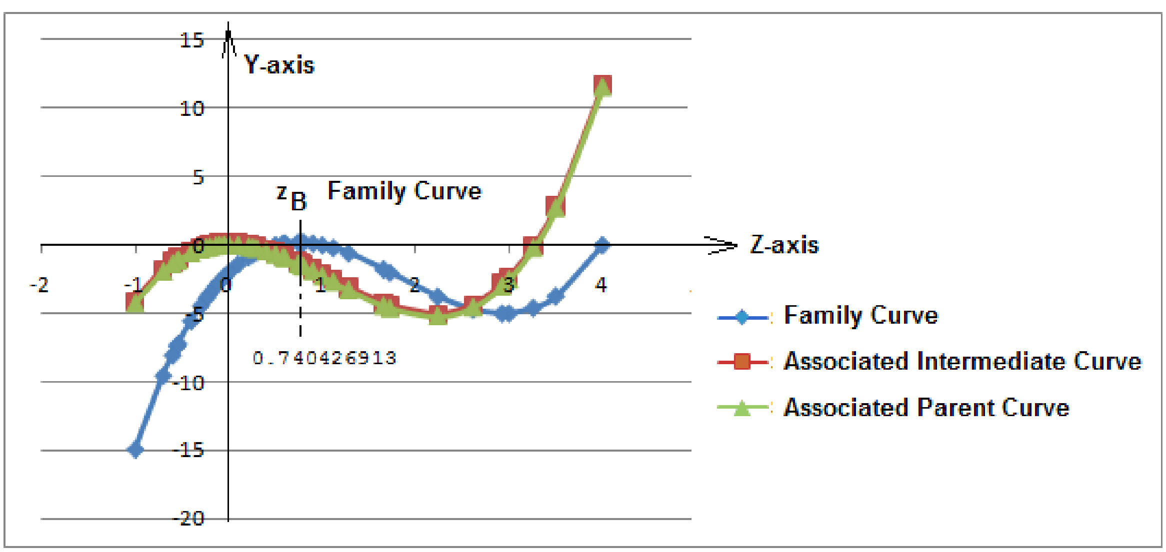
- 3.259573087
- 0.259573087
- -0.240426913

Such roots are determined by subtracting the value z_B for the *Family Curve* given in *Table 27* from these respective root values as follows:

- 4 – 0.740426913 = 3.259573087
- 1 – 0.740426913 = 0.259573087
- ½ – 0.740426913 = -0.240426913

Now the *Associated Intermediate Curve* is displaced horizontally to the left of the *Family Curve* by a distance of z_B (*ref. Figure 27*). This is because z_B for the *Associated Intermediate Curve* aligns with the y-axis. Likewise, since the *Associated Intermediate Curve* is an exact copy of the *Family Curve*, its roots too must be located by the same amount to the left of the respective roots for the *Family Curve*. Hence, the subtraction by an amount z_B = 0.740426913 (*ref. Table 27*) is applied to the roots of the *Family Curve* in order to determine the roots for the *Associated Intermediate Curve*.

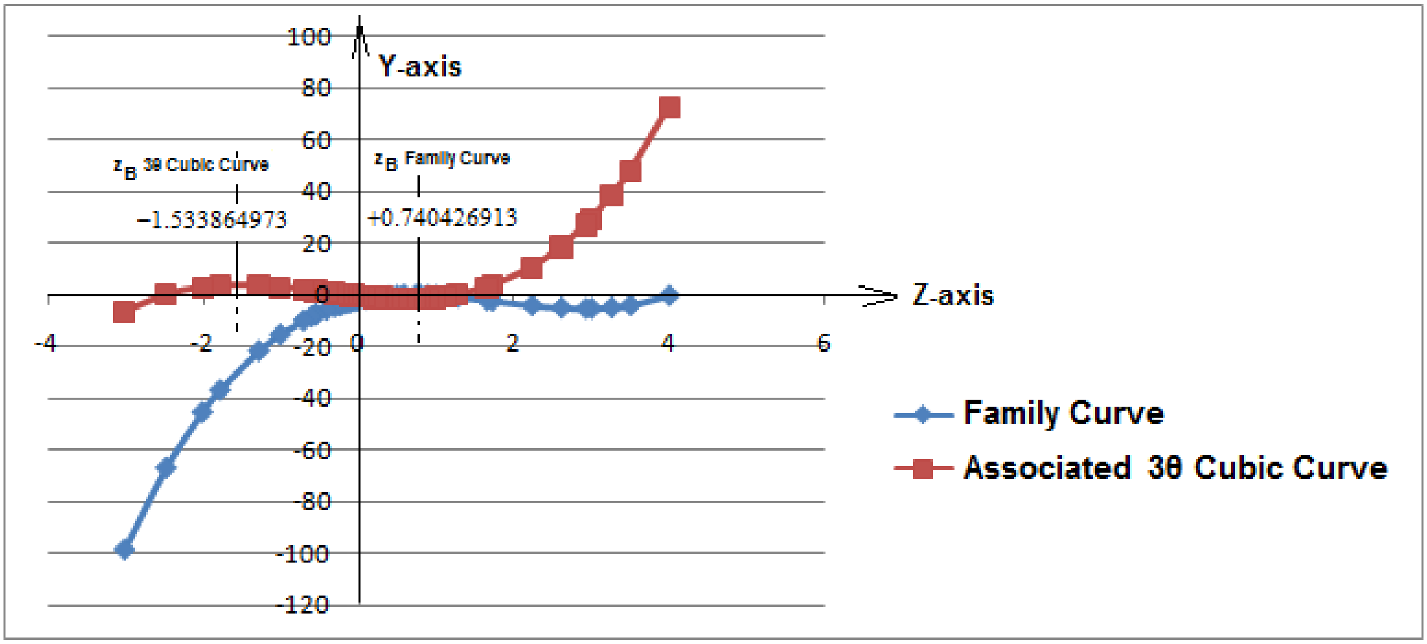
Figure 27. Second Relative Positioning of *Family Curve* to *Other Associated Curves*.



For the *three root values* specified above, respective 'y' values for the *Associated Intermediate Curve* all equal zero *(ref. Table 28)*. Accordingly, respective 'z' values for the *Associated Parent Curve* all must occur at y equals -0.203424598 since this curve is an exact copy of the *Associated Intermediate Curve*, with its only exception being that it *rides* a distance a ν below it.

Horizontal spacing between the *Family Curve* and its *Associated 3θ Cubic Curve* again is related in terms of separation between their relative z_B values *(Ref. Figure 28)*. Since the two curves are identical, each independent set of z_R, z_S, and z_T values for respective elevations on the *Family Curve* is exhibited in exactly the same fashion on the *3θ Cubic Curve*, with the only exception being that on the latter curve, they are displaced horizontally to the left an amount equal to their relative z_B separation. For this analysis, vertical displacement are irrelevant.

Figure 28. Second Relative Positioning of *Family Curve* to *Assoc. 3θ Cubic Curve*.



That is to say, the three respective roots z_R = R = 1, z_S = S = 4, and z_T = T = ½ on the *Family Curve*, repeat themselves upon the *Associated 3θ Cubic Curve*, but at the following z-axis locations *(ref. Table 27 and Figure 26)*:

$$
\begin{aligned}
z_{S'} &= 4 - [(z_B)_{\text{Family Curve}} - (z_B)_{\text{Associated } 3\theta \text{ Cubic Curve}}] \\
&= 4 - [0.740426913 - (-1.533864973)] \\
&= 4 - 2.274291885 \\
&= +1.725708115
\end{aligned}
$$

$$z_{R'} = 1 - [(z_B)_{\text{Family Curve}} - (z_B)_{\text{Associated } 3\theta \text{ Cubic Curve}}]$$
$$= 1 - [0.740426913 - (-1.533864973)]$$
$$= 1 - 2.274291885$$
$$= -1.274291885$$

$$z_{T'} = 1.118033989 - [(z_B)_{\text{Family Curve}} - (z_B)_{\text{Associated } 3\theta \text{ Cubic Curve}}]$$
$$= 1/2 - [0.740426913 - (-1.533864973)]$$
$$= 1/2 - 2.274291885$$
$$= -1.774291885$$

Each of these above three values lies horizontally with respect to one another, validated as follows:

$$z^3+1.322875656z^2-3z-0.440958552=y$$

$$z_{S'}^3+1.322875656z_{S'}^2-3z_{S'}-0.440958552=y$$
$$(1.725708115)^3 + 1.322875656(1.725708115)^2 - 3(1.725708115) - 0.440958552 = y$$
$$3.460808394 = y$$

$$z_{R'}^3+1.322875656z_{R'}^2-3z_{R'}-0.440958552=y$$
$$(-1.274291885)^3 + 1.322875656(-1.274291885)^2 - 3(-1.274291885) - 0.440958552 = y$$
$$3.460808394 = y$$

$$z_{T'}^3+1.322875656z_{T'}^2-3z_{T'}-0.440958552=y$$
$$(-1.774291885)^3 + 1.322875656(-1.774291885)^2 - 3(-1.774291885) - 0.440958552 = y$$
$$3.460808394 = y$$

Equation 36 is applied below in order to check that tan 3θ for the *Associated 3θ Cubic Function* $z^3 + 1.322875656z^2 - 3z - 0.440958552 = y$ indeed is denoted by its last term:

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$\zeta = \frac{-0.440958552 - 1.322875656}{1 - (-3)} = -0.440958552$$

Where,

$$\zeta = -\beta/3$$
$$= -1.322875656/3$$
$$= -0.440958552$$
$$3\theta = 156.2045089^\circ$$
$$\theta = 52.06816963^\circ$$

Then, its three *roots*, as portrayed in *Table 28*, are determined as:

$$\tan\theta = 1.283085371$$
$$\tan(\theta + 120^\circ) = -0.139327752$$
$$\tan(\theta + 240^\circ) = -2.46663327$$

15.2. Regions of *Three Dimensional Space* Governed by 3θ Cubic Functions.

RST Spreads designate *the very realms of three dimensional space which 3θ Cubic Functions* occupy.

So, the *3θ Cubic Function* is established by moving left hand terms in *Equation 22* to the right hand side, and then setting the resulting zero in the equation to 'y' as follows:

$$z^3 = 3z - \zeta(1 - 3z^2) \qquad \textit{[Ref. Equation 22]}$$

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

$$z^3 - 3\zeta z^2 - 3z + \zeta = y$$

Now, the difference between the last two equations is that the *former* holds true only when z equals either z_R, z_S, or z_T, while the *later* encompasses not only these same root set values, but also all intermediate z values in between – Where, such intermediate points may be quantified as:

$$z_a{}^3 - 3\zeta z_a{}^2 - 3z_a + \zeta = y_a$$

$$z_b{}^3 - 3\zeta z_b{}^2 - 3z_b + \zeta = y_b$$

$$z_c{}^3 - 3\zeta z_c{}^2 - 3z_c + \zeta = y_c$$

.

$$z_x{}^3 - 3\zeta z_x{}^2 - 3z_x + \zeta = y_x$$

Upon moving the variable *y* to the left-hand side, a new *family of equations* assumes the following form:

$$z_a{}^3 - 3\zeta z_a{}^2 - 3z_a + (\zeta - y_a) = 0$$

$$z_b{}^3 - 3\zeta z_b{}^2 - 3z_b + (\zeta - y_b) = 0$$

$$z_c{}^3 - 3\zeta z_c{}^2 - 3z_c + (\zeta - y_c) = 0$$

.

$$z_x{}^3 - 3\zeta z_x{}^2 - 3z_x + (\zeta - y_x) = 0$$

Since the right hand side of the equation equals zero, its three *Cubic Equation* roots are delineated as follows, where the first root is to be labeled as z_f:

$$(z - z_f)(z^2 + Mz + N) = 0$$

$$z^3 + (M - z_f)z^2 + (N - Mz_f)z - Nz_f = 0$$

Where,

$$z^3 - 3\zeta z^2 - 3z + (\zeta - y) = 0$$

Comparing respective second and fourth term coefficients from the last two equations yields:

$$M - z_f = -3\zeta$$
$$M = -3\zeta + z_f$$

$$-Nz_f = (\zeta - y)$$
$$N = -\frac{(\zeta - y)}{z_f}$$

Comparing respective third term coefficients between these two given equations above produces the following check:

$$N - Mz_f = -.3$$
$$-\frac{(\zeta - y)}{z_f} - (-3\zeta + z_f)z_f = -.3$$
$$-(\zeta - y) - (-3\zeta + z_f)z_f^{\,2} = -.3z_f$$
$$0 = -3z_f + (\zeta - y) - (3\zeta)z_f^{\,2} + (z_f)z_f^{\,2}$$

Or,

$$0 = z_f^{\,3} - (3\zeta)z_f^{\,2} - 3z_f + (\zeta - y)$$

Then,

$$(z - z_f)(z^2 + Mz + N) = 0$$
$$(z - z_f)[z^2 + (z_f - 3\zeta)z - \frac{(\zeta - y)}{z_f}] = 0$$

Solving the remaining *Quadratic Equation* via *Quadratic Formula* gives the following expression for the other two roots:

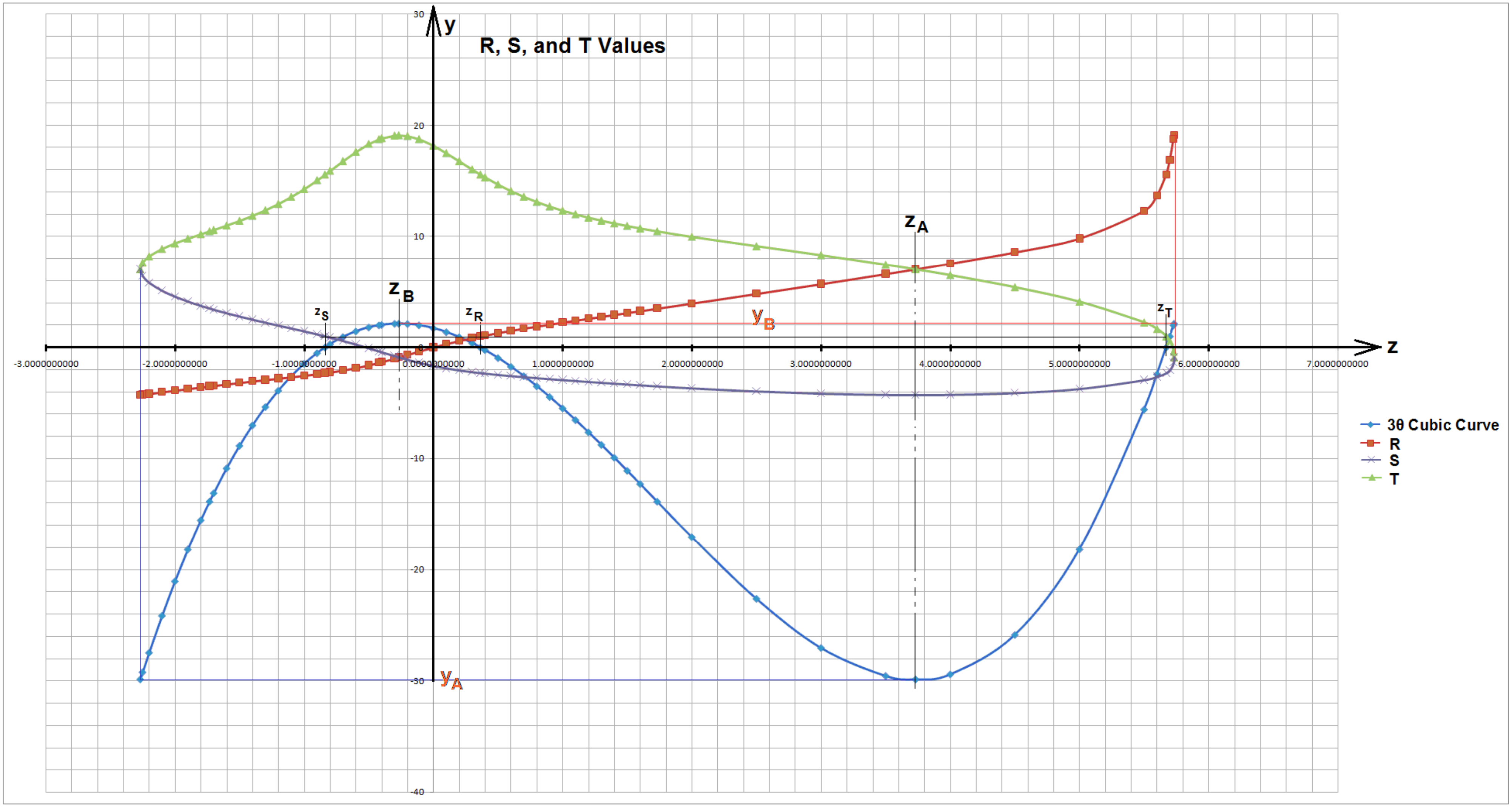
$$z_1, z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} =$$
$$= \frac{-M \pm \sqrt{M^2 - 4N}}{2}$$
$$= \frac{1}{2}[3\zeta - z_f \pm \sqrt{(z_f - 3\zeta)^2 + 4\frac{(\zeta - y)}{z_f}}]$$

Respective ***RST Spreads*** then may be charted for the $z^3 - 3\zeta z^2 - 3z + \zeta = y$ *norm*. This is accomplished by setting:

$$z_R = R\tan\theta = z_f$$
$$z_S = S\tan\theta = z_2$$
$$z_T = T\tan\theta = z_1$$

Figure 29 depicts such ***RST Spread*** for $\zeta = \tan(3\theta) = \tan 60^\circ = \sqrt{3}$.

Figure 29. RST Spread for ζ = tan(3θ) = tan 60° = √3



Most importantly, notice that the *RST Spread* is contained *within a range of z values* bordered by the following:

- On its left, the *z value* which intersects the *3θ Cubic Curve* at y_A and the slope of the *Curve* is not equal to zero
- On its right, the *z value* which intersects the *3θ Cubic Curve* at y_B and the slope of the *Curve* is not equal to zero

Upon further inspection, notice that such boundary specifies the region along the *3θ Cubic Curve* where *three real roots exist*! That is:

- Below y_A, and to the left of this *real root region*, only one value on the *3θ Cubic Curve* exists for each drop in elevation
- Above y_B, and to the right of this *real root region*, only one value on the *3θ Cubic Curve* exists for each rise in elevation

Hence, this *real root region* is bounded below by the y_A horizontal offset as it extends to the left until it intersects the point of non-zero slope on the *3θ Cubic Curve*; and is bounded above by the y_B horizontal offset as it extends to the right until it intersects another point of non-zero slope on the *3θ Cubic Curve.*

The S and T *Curves* connect, or interconnect, at these two *real three dimensional space boundaries*, or junctures. The upper *S and T Curve* interconnection point appearing at the left boundary resides at the same elevation, or ordinate, as the *R and T Curve* intersection point; while the lower *S and T Curve* interconnection point lies on the same elevation as the *R and S Curve* intersection point. Moreover, the *R and T Curve* intersection point occurs at z_A; while the *R and S Curve* intersection point occurs at Z_B.

The R curve has a value of unity at $z = z_R = \tan\theta$. The resulting horizontal offset intersects the remaining two S and T Curves at:

- $z = z_S = S \tan \theta = \tan (120^\circ + \theta)$, and at
- $z = z_T = T \tan \theta = \tan (240^\circ + \theta)$.

For the *3θ Cubic Function* $z^3 - 3\sqrt{3}z^2 - 3z + \sqrt{3} = y$ *(Ref. Equation 25)*, an ***associated norm volume*** consists of the product between S and T since R equals unity. This volume equates to:

$$S = \frac{\tan(120^o + \theta)}{\tan\theta} \qquad\qquad T = \frac{\tan(240^o + \theta)}{\tan\theta}$$

$$= \frac{\tan(140^o)}{\tan 20^o} \qquad\qquad = \frac{\tan(260^o)}{\tan 20^o}$$

$$= -2.305407289 \qquad\qquad = 15.58171874$$

$$RST = (1)(-2.305407289)(15.58171874) = -35.92220796$$

Now, a new attribute is described more fully in *Figure 30*; wherein it is postulated that *RST Spreads* possess an ***inherent capability of interchangeability.***

As depicted, each of the three *R, S, and T Curves* intersect a vertical line drawn though point z_R at the following respective ordinate values:

- $y_R = R = 1$
- $y_S = S = -2.305407289$
- $y_T = T = 15.58171874$

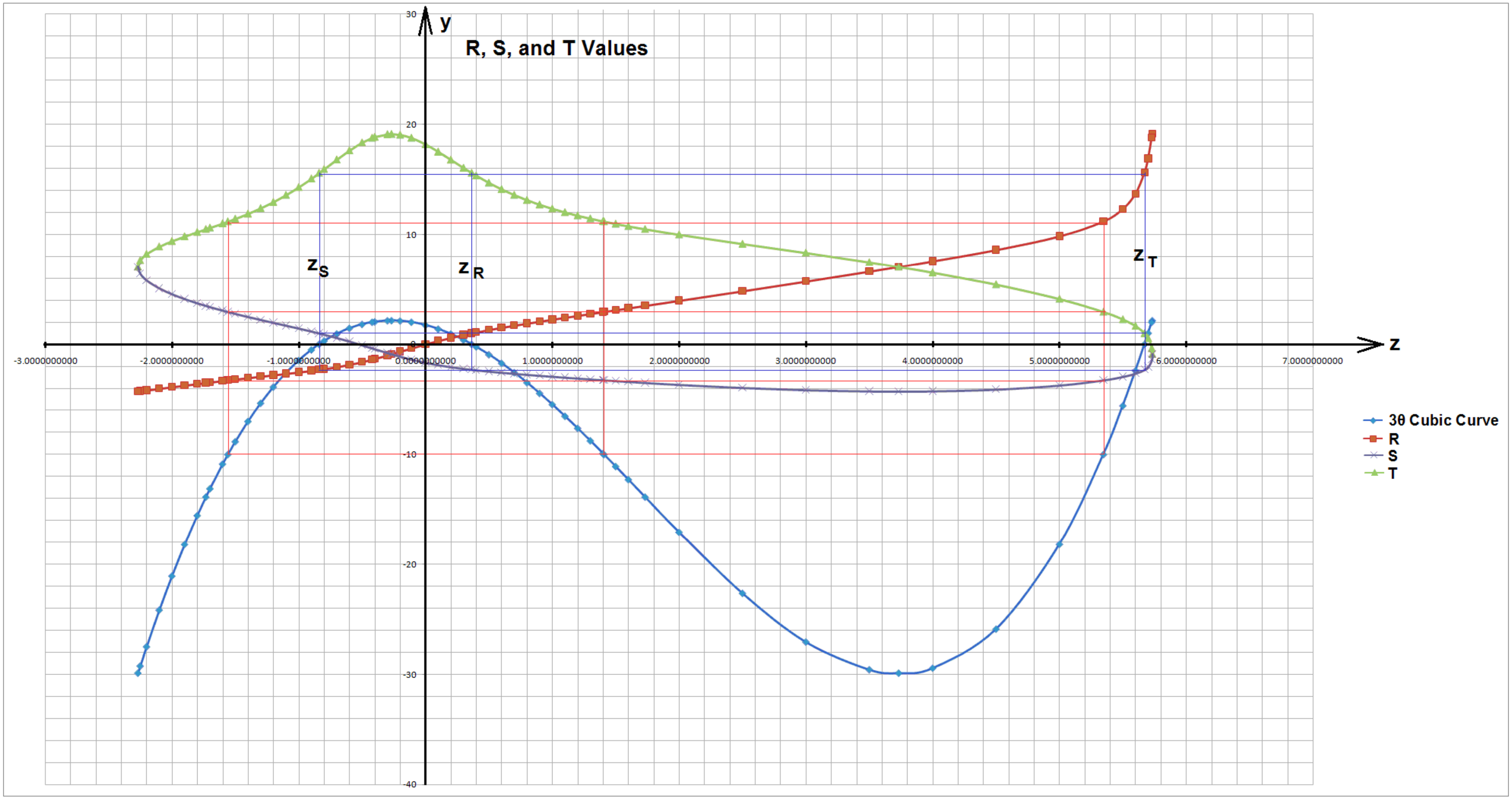
Notice that the *horizontal offset* of $y_R = 1$ also intersects *Curve S* at z_S, and *Curve T* at z_T. The offset $y_S = -2.305407289$ also intersects *Curve R* at z_S, and *Curve S* at z_T. Lastly, the offset $y_T = 15.58171874$ also intersects *Curve R* at z_T, and *Curve T* at z_S.

In other words, *R, S, and T Curves* continuously cross these elevations at respective z_R, z_S, and z_T emplacements, thereby ***demonstrating interchangeability.***

Such affinity for *inherent interchangeability* also exists with respect to *other* **elevations** upon the *norm* (*ref. Section 15.1*). *Figure 30* emphasizes this for an elevation of y = -10 on the *3θ Cubic Curve* where it assumes the following three z values:

- $z_{R'} = 1.407843815$
- $z_{S'} = -1.558550052$
- $z_{T'} = 5.3468586598$

Figure 30. RST Spread Interchangeability.



Vertical extensions drawn through such newly determined $z_{R'}$, $z_{S'}$, and $z_{T'}$ values intersect the *R, S, and T Curves* at the three following respective vertical locations:

- y_1 = 11.17554923 = R, T, T
- y_2 = 2.942555408 = R, S, T
- y_3 = -3.257548768 = R, S, S

For example, the above notation indicates that a y_1 value 11.17554923 occurs on the *R Curve* when $z_{R'}$ = 5.3468586598, and also cuts off the *T Curve* at two locations when $z_{T'}$ = 1.407843815 and -1.558550052. Again this ***demonstrates interchangeability*** with the respective cutoffs along the *3θ Cubic Curve* when y is equal to -10, with the only difference being a swapping of respective R, S, and T Curves. This is easily observed by comparing the results in this paragraph against those $z_{R'}$, $z_{S'}$, and $z_{T'}$ values determined for y = -10 above.

Table 29 gives the associated plot points for *Figure 30* (as well as *Figure 29*). Its first column provides *unique designators* for z-values which exhibit ***interchangeability attributes***. As shown:

- z_{1-1} thru z_{1-3} result in values of y = 2 (ref. 3rd column)
- z_{2-1} thru z_{2-3} result in values of y = -10
- z_R, z_S, and z_T result in values of y = 0
- z_A returns a value of y = -29.85640646
- $z_{\text{Left Boundary}}$ ($z_{L/B}$) returns a value of y = -29.85640646
- z_B returns a value of y = 2.143593539
- $z_{\text{Right Boundary}}$ ($z_{R/B}$) returns a value of y = 2.143593539

For the *3θ Cubic Function* described in column 3 of *Table 29*, since ζ = tan 60°, associated z-values are given in columns 4 and 5 for every z = z_f value which is specified in column 2. Column 4 and 5 values are calculated using the equation previously developed above and reprinted below of:

$$z_1, z_2 = \frac{1}{2}[3\zeta - z_f \pm \sqrt{(z_f - 3\zeta)^2 + 4\frac{(\zeta - y)}{z_f}}]$$

For every ***elevation*** on the *3θ Cubic Curve* within the *real root region*, three associated z-values exist as z = z_f, z_1, and z_2. *Each set of these values* also represents *roots* for the *3θ Cubic Curve* were the origin to be placed at that particular ***elevation***. Furthermore, the *root values* for each *set of roots* repeat themselves two more additional times, in various combinations. This is demonstrated in *Table 29* for the group sets highlighted above of z_{1-1} thru

z_{1-3}, z_{2-1} thru z_{2-3}, z_R, z_S, and z_T, z_A and $z_{L/B}$, z_B and $z_{R/B}$. *For example*, z_{2-1} thru z_{2-3} all share identical root sets of 5.3468586598, 1.407843815, and −1.558550082, respectively.

In order to develop *Table 29*, a determination of tan (3θ′) is needed. From this, values for the 3θ′ column may be ascertained. Thereafter, θ′ column values represent 1/3 of the 3θ′ column values, and tan θ′ values are calculated from this.

The tan (3θ′) values are derived as follows from the $z^3 - 3\zeta z^2 - 3z + (\zeta - y) = 0$ *Cubic Equation*:

Where,

$$\zeta' = \frac{\delta' - \beta'}{1 - \gamma'}.$$

$$= \frac{(\zeta - y) - (-3\zeta)}{1 - (-3)}$$

$$= \frac{4\zeta - y}{4}$$

$$= \zeta - \frac{y}{4}$$

Since R = z_f/(tan θ′), S = z_2/(tan θ′), and T = z_1/(tan θ′), considering that tan θ′ remains constant for each value of z = z_f selected, RST values for every elevation within the *real root region* must also repeat themselves three successive times. This too is validated by *Table 29*.

Lastly, z_1 and z_2 values inserted into *Table 29* for z_f = 0 are calculated below:

$$z_f^{\ 3} - 3\zeta zf^{\ 2} - 3z_f + \zeta = y$$

$$0^3 - 3\zeta(0)^2 - 3(0) + \zeta = y$$

$$\zeta = y$$

$$z^3 - 3\zeta z^2 - 3z \ + \zeta = y$$

$$z^3 - 3\zeta z^2 - 3z \ + \zeta = \zeta$$

$$z^3 - 3\zeta z^2 - 3z = 0$$

$$z^2 - 3\zeta z - 3 = 0$$

Table 29. Plot Points for *Figure 30*.

FOR 3θ= 60°			ROOTS/z-VALUES along with $z = z_f$		θ FUNCTIONS			RST FUNCTIONS		
DES	$z = z_f$	$y = z^3 - 3\zeta z^2 - 3z + \zeta$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	**R**	**S**	**T**
$z_{R/B}$	5.732050809	2.143593539	-0.267949193	-0.267949193	50.10390912	16.70130304	0.30003917	19.10434181	-0.89304738	-0.893047382
z_{1-1}	5.728056783	2	-0.111187241	-0.420717118	50.93532161	16.97844054	0.30531927	18.76087519	-1.37795794	-0.364167123
	5.70	1.002058594	0.185722649	-0.689570227	55.98163058	18.66054353	0.33771396	16.8781888	-2.04187658	0.549940691
z_T	5.671281820	0	0.363970234	-0.839099631	60	20	0.36397023	15.58171874	-2.30540729	1
	5.60	-2.403289169	0.680814442	-1.084662019	66.79731699	22.26577233	0.40943219	13.67747854	-2.64918598	1.662825878
	5.50	-5.576559979	1.010796798	-1.314644375	72.26166376	24.08722125	0.44705398	12.30276474	-2.94068372	2.261017309
z_{2-1}	5.346858660	-10	1.407843815	-1.558550052	76.70531381	25.56843794	0.47844258	11.17554923	-3.25754877	2.942555408
	5.00	-18.17175976	2.095670028	-1.899517605	80.94532504	26.98177501	0.50912485	9.820773861	-3.73094657	4.116220287
	4.50	-25.86503575	2.848844683	-2.152692261	83.04562307	27.68187436	0.52460825	8.577829204	-4.10342812	5.430422917
	4.00	-29.40638796	3.45154863	-2.255396208	83.71772366	27.90590789	0.52960477	7.552802049	-4.25864027	6.517215891
z_A	3.732050808	-29.85640646	3.732050808	-2.267949192	83.79397689	27.93132563	0.53017296	**7.039308147**	-4.27775345	**7.039308147**
	3.50	-29.54581637	3.955449176	-2.259296753	83.74154785	27.91384928	0.52978226	6.606487688	-4.26457605	7.466178937
	3.00	-27.033321	4.383533354	-2.187380931	83.2826309	27.76087697	0.52636786	5.699436094	-4.15561261	8.327889406
	2.50	-22.61890183	4.747736566	-2.051584144	82.29033644	27.43011215	0.51901771	4.816791282	-3.95282105	9.14754244
	2.00	-17.05255888	5.054396859	-1.858244437	80.53022435	26.84340812	0.50608764	3.951884683	-3.67178386	9.987196765
	1.73	-13.85640646	5.196152423	-1.732050808	79.10660535	26.36886845	0.49572726	3.493959207	-3.49395921	10.48187762
	1.60	-12.27409939	5.260289656	-1.664137234	78.2330829	26.0776943	0.4894123	3.269227218	-3.40027671	10.74817632
	1.50	-11.08429214	5.306343743	-1.61019132	77.47960945	25.82653648	0.4839904	3.099235025	-3.32690756	10.96373759
z_{2-2}	1.407843815	-10	5.34685866	-1.558550052	76.70531381	25.56843794	0.47844258	2.942555408	-3.25754877	11.17554923
	1.40	-9.908407941	5.350221442	-1.55406902	76.63557827	25.54519276	0.47794411	2.929212805	-3.25157062	11.19424083
	1.30	-8.752446787	5.391911149	-1.495758726	75.68952761	25.22984254	0.47120062	2.758909782	-3.17435645	11.44292032
	1.20	-7.622408681	5.431396973	-1.43524455	74.62894096	24.87631365	0.46368216	2.587979653	-3.09531974	11.71362071
	1.10	-6.524293624	5.468658593	-1.372506171	73.44054532	24.48018177	0.45530859	2.415943873	-3.01445261	12.01088384
	1.00	-5.464101615	5.503671051	-1.307518628	72.11090878	24.03696959	0.44600205	2.242142141	-2.93164262	12.34001279
	0.90	-4.447832655	5.5364045	-1.240252077	70.62749299	23.54249766	0.43569462	2.065667024	-2.84660869	12.70707579
	0.80	-3.481486743	5.566823897	-1.170671474	68.98036089	22.99345363	0.42433998	1.885280757	-2.7588055	13.11878247

FOR 3θ= 60°			ROOTS/z-VALUES along with z = z_f		θ FUNCTIONS			RST FUNCTIONS		
DES	$z = z_f$	$y = z^3 - 3\zeta z^2 - 3z + \zeta$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T
	0.70	-2.57106388	5.594888627	-1.098736205	67.16476486	22.38825495	0.41193046	1.699315935	-2.66728563	13.58211914
	0.60	-1.722564065	5.620552054	-1.024399632	65.18480519	21.7282684	0.39851995	1.505570786	-2.57051026	14.10356496
	0.50	-0.941987298	5.643760982	-0.947608559	63.05818671	21.01939557	0.38425248	1.301227766	-2.46610914	14.68763699
	0.40	-0.23533358	5.664455014	-0.868302592	60.82173434	20.27391145	0.36939368	1.082855549	-2.3506157	15.33446637
z_R	0.363970234	0	5.67128182	-0.839099632	60	20	0.36397023	1	-2.30540729	15.58171874
	0.40	-0.23533358	5.664455014	-0.868302592	60.82173434	20.27391145	0.36939368	1.082855549	-2.3506157	15.33446637
	0.30	0.39139709	5.682565794	-0.786413372	58.53671793	19.51223931	0.35435899	0.846599084	-2.21925613	16.03618332
	0.20	0.932204711	5.698016092	-0.70186367	56.29228833	18.76409611	0.33972865	0.588705125	-2.0659537	16.77225638
	0.10	1.381089283	5.710718717	-0.614566295	54.20478884	18.06826295	0.3262374	0.306525247	-1.88380085	17.50479467
	0.00	1.732050808	5.720575211	-0.524422787	52.41091053	17.47030351	0.31472905	0	-1.66626748	18.17619039
z_{1-2}	-0.111187241	2	5.728056782	-0.420717119	50.93532161	16.97844054	0.30531927	-0.36416712	-1.37795795	18.76087519
	-0.20	2.116204711	5.731289814	-0.335137391	50.26476205	16.75492068	0.3010595	-0.6643205	-1.1131932	19.03706662
z_B	-0.267949192	2.143593539	5.732050808	-0.267949193	50.10390936	16.70130312	0.30003917	-0.89304737	-0.89304738	19.10434171
	-0.30	2.13739709	5.731878674	-0.235726251	50.14039558	16.71346519	0.30027056	-0.99909894	-0.78504616	19.08904641
	-0.40	2.03666642	5.729077664	-0.132925241	50.72579723	16.90859908	0.30398717	-1.31584499	-0.43727253	18.84644539
z_{1-3}	-0.420717119	2	5.728056782	-0.111187241	50.93532161	16.97844054	0.30531927	-1.37795795	-0.36416712	18.76087519
	-0.50	1.808012702	5.722699994	-0.026547572	52.00230192	17.33410064	0.31211834	-1.60195647	-0.08505611	18.33503259
	-0.60	1.445435935	5.712530736	0.083621686	53.88705792	17.96235264	0.32419341	-1.85074705	0.257937649	17.62074899
	-0.70	0.94293612	5.698321046	0.197831376	56.24488867	18.74829622	0.3394211	-2.06233498	0.582849383	16.78835261
	-0.80	0.294513257	5.679780712	0.31637171	58.91075174	19.63691725	0.35681017	-2.24208853	0.886666728	15.91821397
z_S	-0.839099631	0	5.671281819	0.363970234	60	20	0.36397023	-2.30540729	1	15.58171874
	-0.90	-0.505832655	5.656568383	0.43958404	61.71682758	20.57227586	0.37532315	-2.39793363	1.171214836	15.07119507
	-1.00	-1.464101615	5.628278533	0.56787389	64.51626541	21.5054218	0.39401979	-2.53794358	1.441231894	14.28425338
	-1.10	-2.586293624	5.594423727	0.701728696	67.19757531	22.39919177	0.41215375	-2.6689069	1.702589599	13.57363282
	-1.20	-3.878408681	5.55440992	0.841742503	69.68828147	23.22942716	0.42920863	-2.79584313	1.961149996	12.94104902
	-1.30	-5.346446787	5.507501122	0.988651301	71.95051239	23.98350413	0.44488375	-2.92211167	2.222268849	12.379641
	-1.40	-6.996407941	5.452767211	1.143385212	73.9726971	24.6575657	0.45905162	-3.04976596	2.490755217	11.87833132

FOR 3θ= 60°			ROOTS/z-VALUES along with $z = z_f$		θ FUNCTIONS			RST FUNCTIONS		
DES	$z = z_f$	$y = z^3 - 3\zeta z^2 - 3z + \zeta$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T
	-1.50	-8.834292144	5.389003864	1.307148559	75.76080412	25.25360137	0.47170746	-3.179937	2.771100045	11.42446186
z_{2-3}	-1.558550052	-10	5.34685866	1.407843815	76.70531381	25.56843794	0.47844258	-3.25754877	2.942555408	11.17554923
	-1.60	-10.86609939	5.314603817	1.481548606	77.33101201	25.777004	0.48292383	-3.31315186	3.067872203	11.00505596
	-1.70	-13.09782969	5.227337418	1.668815005	78.70439168	26.23479723	0.4928156	-3.44956611	3.386286872	10.60708588
	-1.73	-13.85640646	5.196152423	1.732050808	79.10660535	26.36886845	0.49572726	-3.49395921	3.493959207	10.48187762
	-1.80	-15.53548304	5.123949152	1.872203271	79.9034412	26.6344804	0.50151563	-3.58912044	3.733090575	10.21692814
	-1.84	-16.54589505	5.078127039	1.957125015	80.3296492	26.77654973	0.50462274	-3.6445041	3.878392463	10.06321491
	-1.90	-18.18505944	4.999336077	2.096816345	80.9500409	26.98334697	0.5091594	-3.73164085	4.118192382	9.818803536
	-2.00	-21.05255888	4.844610438	2.351541985	81.86438268	27.28812756	0.5158761	-3.87689989	4.558346432	9.391034838
	-2.10	-24.14398138	4.641323389	2.654829034	82.6645149	27.5548383	0.52178413	-4.02465285	5.087983449	8.89510258
	-2.20	-27.46532692	4.333868894	3.062283529	83.36624885	27.78874962	0.52698927	-4.17465801	5.810902944	8.223827505
	-2.25	-29.21409583	4.050877255	3.395275167	83.6845687	27.89485623	0.52935781	-4.2504332	6.413951248	7.652436974
$z_{L/B}$	-2.267949192	-29.85640646	3.732101667	3.731999947	83.79397688	27.93132563	0.53017296	-4.27775345	7.039212216	7.039404078

Applying the Quadratic Formula renders:

$$z_1, z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3\zeta \pm \sqrt{(-3\zeta)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{1}{2}[3\zeta \pm \sqrt{9\zeta^2 + 12}]$$

$$= \frac{1}{2}[3\zeta \pm \sqrt{9(3) + 12}]$$

$$= \frac{1}{2}(3\zeta \pm \sqrt{39})$$

$$= \frac{1}{2}(5.196152423 \pm 6.244997998)$$

$$= 5.720575211, -0.524422787$$

Additional attributes are represented as *coefficients* of the *3θ Cubic Function* when $\zeta = \sqrt{3}$ as follows:

$$z_1^{\ 3} - 3(z_1) + \sqrt{3}(1 - 3z_1^{\ 2}) = y \qquad \textit{[Ref. Equation 25]}$$

$$z^3 - 3\sqrt{3}z^2 - 3z + \sqrt{3} = y$$

These *coefficients*, or new *attributes*, may be described by the following *symbols* once *like terms* become compared to those appearing in the *Generalized Cubic Function*:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = y \qquad \textit{[Ref. Equation 32]}$$

Such that

$$\alpha = 1$$

$$\beta = -(z_f + z_1 + z_2) = -3\sqrt{3} = -5.196152423$$

$$\gamma = -(z_f z_1 + z_f z_2 + z_1 z_2) = -3$$

$$\delta = -z_R z_S z_T = \sqrt{3}$$

Notice that both *coefficients* β and γ remain *constant* no matter what values of z_f are applied within the *real root region*.

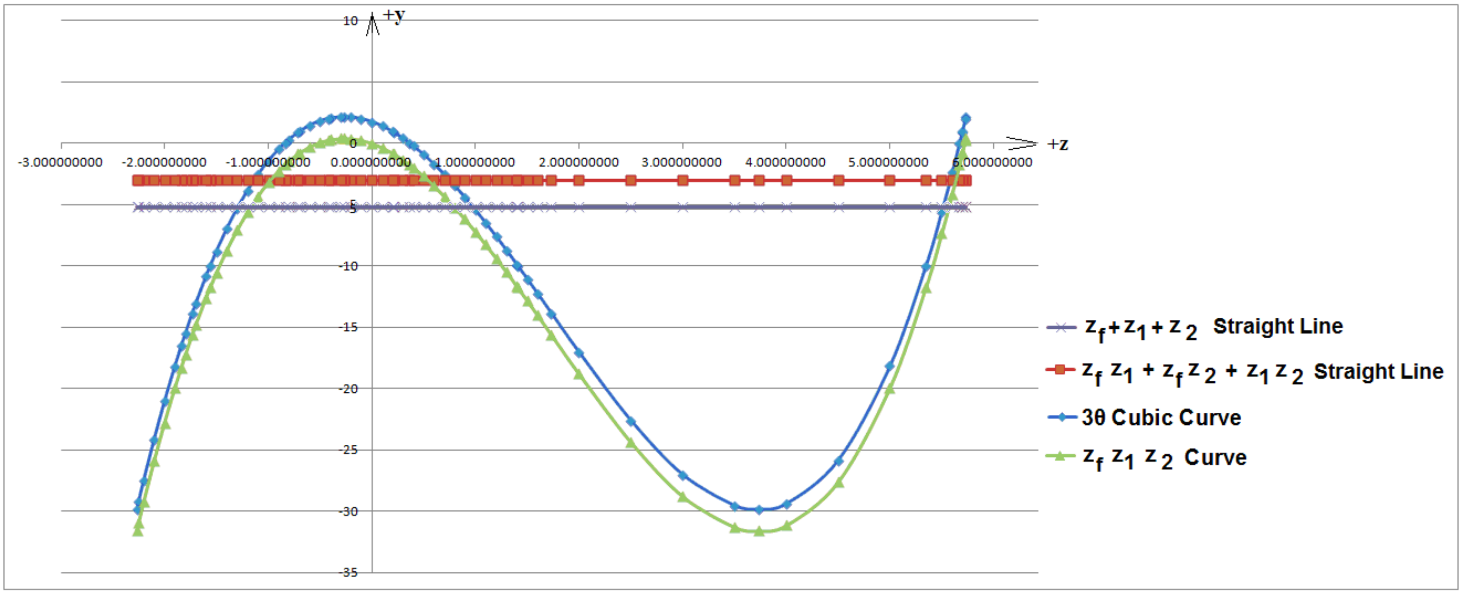
Figure 31 reflects this consideration by charting three functions as follows, with respect to the *3θ Cubic Curve* when $\zeta = \sqrt{3}$, the first two of which appear as *straight lines*:

$$-(z_f + z_1 + z_2) = -5.196152423 = y_\beta$$

$$-(z_f z_1 + z_f z_2 + z_1 z_2) = -3 = y_\gamma$$

$$z_f z_1 z_2 = -\delta' = y_{\delta'}$$

Figure 31. Known Attribute Plot of the *ζ = √3 3θ Cubic Curve.*



The above relationships hold for all z_f, z_1, and z_2 values within the *real root region*. This also applies for the roots z_R, z_S, and z_T, when y = 0.

Now for the *3θ Cubic Function* when $\zeta = \sqrt{3}$:

$$z^3 - 3\sqrt{3}z^2 - 3z + \sqrt{3} = y$$

As $z = z_f$:

$$z_f^{\ 3} - 3\sqrt{3}z_f^{\ 2} - 3z_f + \sqrt{3} = y$$

$$z_f^{\ 3} - 3\sqrt{3}z_f^{\ 2} - 3z_f + (\sqrt{3} - y) = 0$$

$$z_f^{\ 3} + \beta z_f^{\ 2} + \gamma z_f + \delta' = 0$$

Such that,

$$\delta' = -z_f z_1 z_2$$

Then,

$$-\delta' = z_f z_1 z_2 = y - \zeta = y - \sqrt{3} = y_{\delta'}$$

And, at y = 0,

$$-\delta = z_R z_S z_T = -\sqrt{3}$$

As described above, the $y_{\delta'}$ *Cubic Function* is of identical shape to the *3θ Cubic Curve* when $\zeta = \sqrt{3}$, with the only exception being that it resides a distance of $\sqrt{3}$ below it.

Such contention is verified, starting with the *3θ Cubic Function* when $\zeta = \sqrt{3}$ as follows:

$$y = z_f^{\ 3} - 3\sqrt{3}z_f^{\ 2} - 3z_f + \sqrt{3}$$

$$y - \sqrt{3} = z_f^{\ 3} - 3\sqrt{3}z_f^{\ 2} - 3z_f + \sqrt{3} - \sqrt{3}$$

$$= z_f^{\ 3} - 3\sqrt{3}z_f^{\ 2} - 3z_f$$

$$= z_f z_1 z_2$$

$$= y_{\delta'}$$

So this *new Cubic Function*, also portrayed in *Figure 31*, is depicted as 'riding' a distance of $\zeta = \sqrt{3}$ below the established, or given *3θ Cubic Curve*.

Accordingly, *Figure 31* displays a *Cubic Function* whose ordinate values respectively equal the product of any $z_f z_1 z_2$ ***Spread*** which appears in the *real root region* on the *3θ Cubic Curve*. For example for any of the z_{2-1} thru z_{2-3} listings in *Table 29*, $y = -10$. Then $-10-\sqrt{3}$, which equals -11.732050808, also represents the product of the roots $z_f z_1 z_2$, regardless of which order they are presented in, calculated as follows:

$$z_f z_1 z_2 = [5.3468586598][1.407843815][-1.558550082]$$
$$= -11.7320808$$

And, it is precisely this -11.732050808 value which appears in *Figure 31* as the *ordinate value* for the *new Cubic Function* at any of the $z_f = 5.3468586598$, $z_1 = 1.407843815$, or $z_2 = -1.558550082$ coordinates itemized directly above.

Therefore, <u>it is concluded</u> that the *new Cubic Function* presented in *Figure 31* portrays respective ordinates that represent associated ***volumetric depictions in linear fashion*** which truly correspond to the product of any of the *z_f, z_1, and z_2* ***Spreads*** which reside in the *real root region* on the *3θ Cubic Curve*, regardless of elevation.

Figure 32 represents sixteen *3θ Cubic Functions* which exhibit various *arbitrarily selected* 3θ values.

Figure 33 portrays associated R Values for the various *3θ Cubic Curves* presented in *Figure 32*. For any ordinate selected, representing a <u>constant value</u> for R, *Figure 33* illustrates the variability in *3θ Cubic Functions* that is necessary in order to retain such constancy while moving to the right, or increasing in z value. The z-axis depicts a range between $-\sqrt{3}$ thru $+\sqrt{3}$ representing thresholds below and above which *S and T values*, respectively, *start* becoming imaginary. This is in stark contrast to *R values*, as plotted on the y-axis, which remain real from negative infinity thru positive infinity.

Thresholds where respective S and T values *start* becoming *imaginary* are determined by identifying associated z_f values for *3θ Cubic Curve* non-zero slope y_A and y_B values as follows (*ref. Figure 29*):

Where,

$$z_{A,B} = \frac{1}{3}[-\beta' \pm \sqrt{\beta'^2 - 3\gamma'}]$$

For the *3θ Cubic Equation:*

$$\beta' = -3\zeta$$

$$\gamma' = -3$$

$$z_{A,B} = \frac{1}{3}[3\zeta \pm \sqrt{9\zeta^2 + 9}]$$

$$= \zeta \pm \sqrt{\zeta^2 + 1}$$

$$y_{A,B} = z_{A,B}{}^3 - 3\zeta z_{A,B}{}^2 - 3z_{A,B} + \zeta$$

$$= (\zeta \pm \sqrt{\zeta^2+1})^3 - 3\zeta(\zeta \pm \sqrt{\zeta^2+1})^2 - 3(\zeta \pm \sqrt{\zeta^2+1}) + \zeta$$

$$= \zeta^3 \pm 3\zeta^2\sqrt{\zeta^2+1} + 3\zeta(\zeta^2+1) \pm (\zeta^2+1)\sqrt{\zeta^2+1} - 3\zeta(\zeta^2 \pm 2\zeta\sqrt{\zeta^2+1} + \zeta^2 + 1) - 3(\zeta \pm \sqrt{\zeta^2+1}) + \zeta$$

$$= -2\zeta^3 - 2\zeta \pm \sqrt{\zeta^2+1}(3\zeta^2 + \zeta^2 + 1 - 6\zeta^2 - 3)$$

$$= -2\zeta^3 - 2\zeta \pm (-2\zeta^2 - 2)\sqrt{\zeta^2+1}$$

$$= -2(\zeta^2+1)(\zeta \pm \sqrt{\zeta^2+1})$$

Table 30 depicts calculations for determining y_A and y_B values from this above equation.

In *Table 30,* notice that all $z_{B\text{-RIGHT}}$ values are either greater to or equal to $\sqrt{3}$; likewise, all $Z_{A\text{-LEFT}}$ values are either less than or equal to $-\sqrt{3}$. Such values are obtained in the same manner as they were determined in *Table 29*. In this case, the $z_{B\text{-RIGHT}}$ and $Z_{A\text{-LEFT}}$ designations are analogous to respective $z_{\text{Right Boundary}}$ ($z_{R/B}$) $z_{\text{Left Boundary}}$ ($z_{L/B}$) notations rendered in the first column of *Table 29*.

Figure 32. $z^3 - 3\zeta z^2 - 3z + \zeta = y$ ***Curve Set*** **for various Values of** $\zeta = \tan(3\theta)$**.**

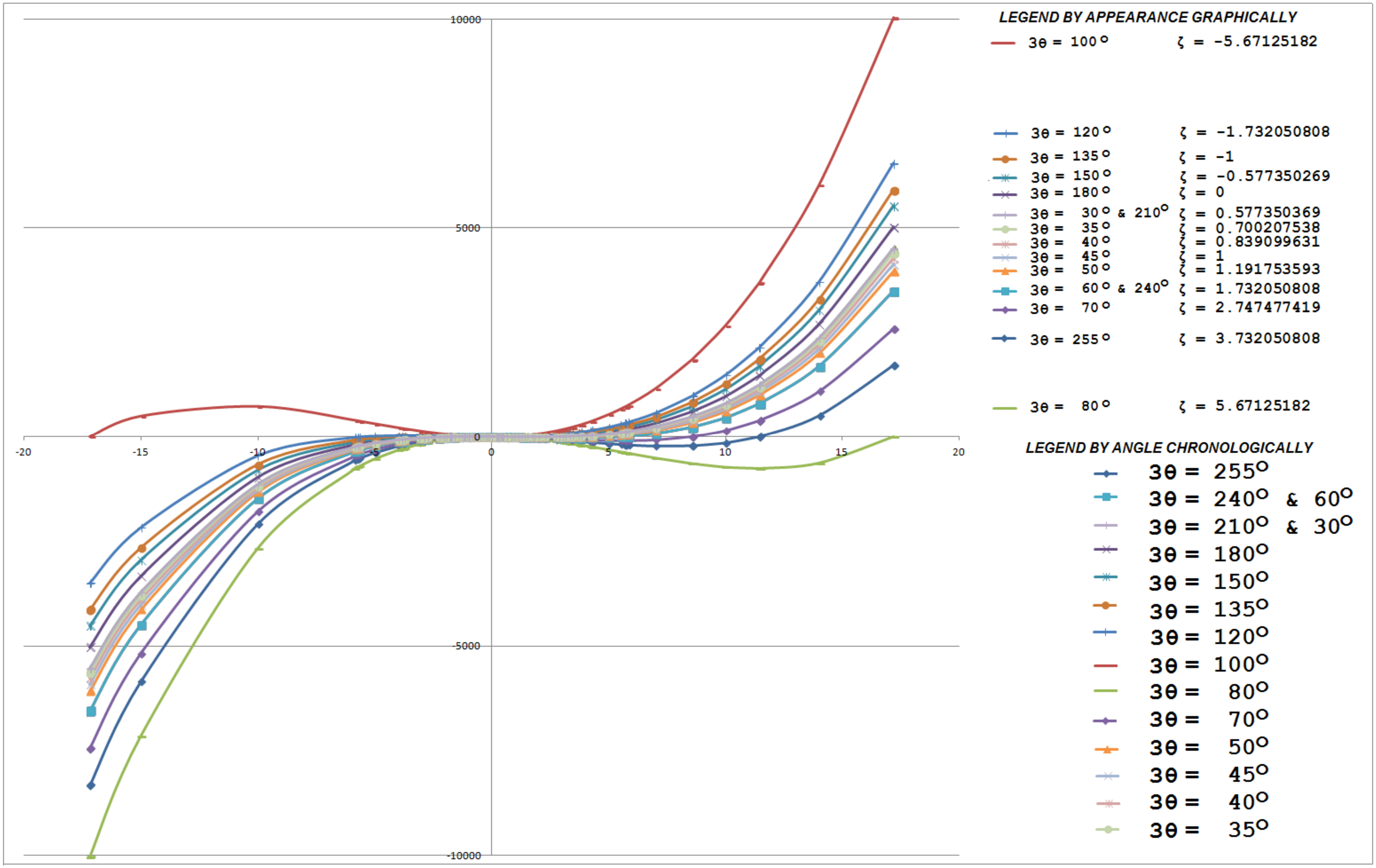


Figure 33. R Values for *3θ Cubic Functions* varying from 255° thru 30°.

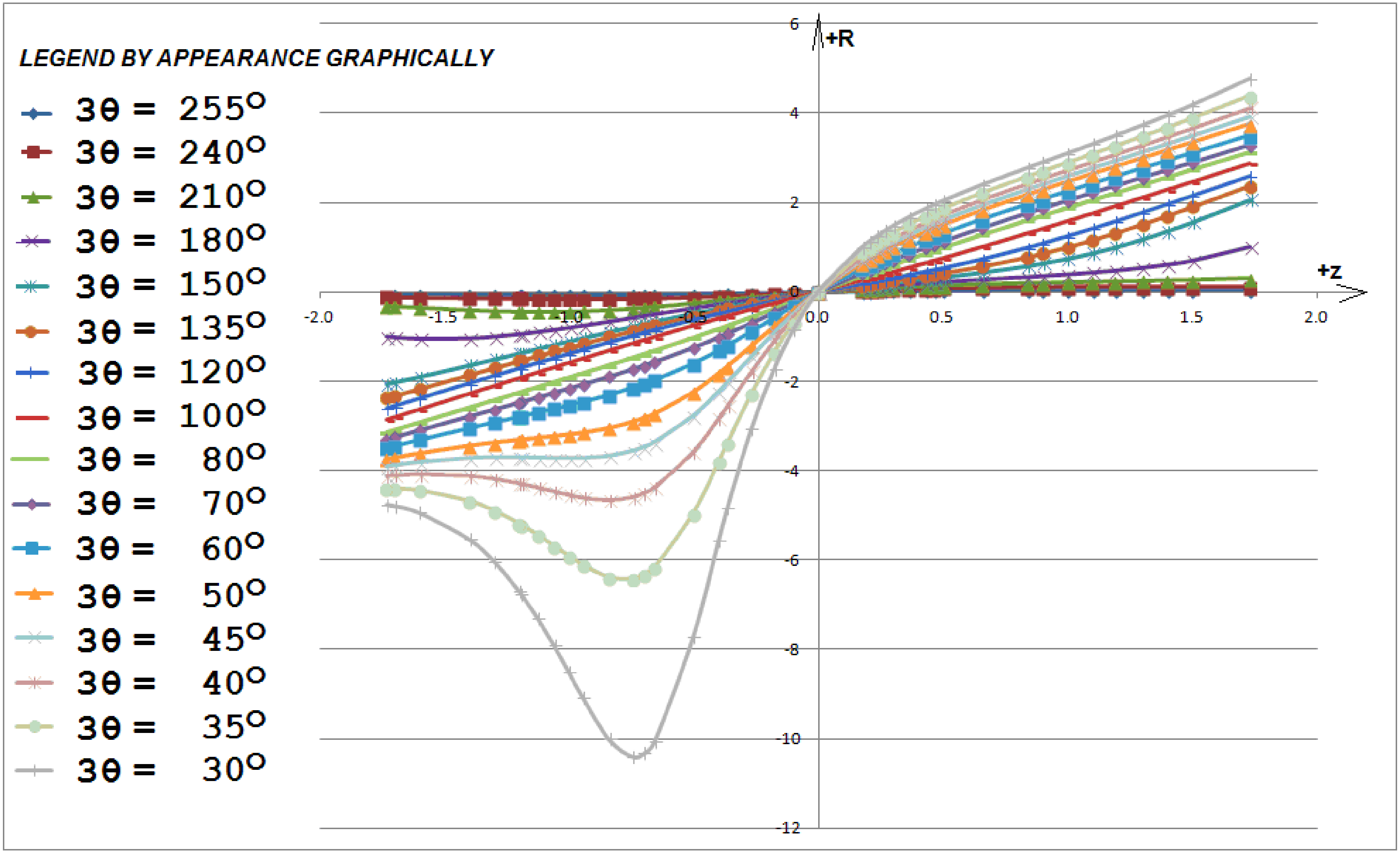


Table 30. Associated z_B and z_A Values for 3θ Cubic Curve Set.

3θ	$\tan(3\theta) = \zeta$	ζ^2+1	$-2(\zeta^2+1)$	$\zeta-\sqrt{\zeta^2+1}$	$\zeta+\sqrt{\zeta^2+1}$	y_B	y_A	$z_{B-RIGHT}$	z_{A-LEFT}
255	3.732050808	14.92820323	-29.85640646	-0.131652498	7.595754113	3.93067048	-226.7819222	11.45945741782	-3.9953558031
240	1.732050808	4	-8	-0.267949192	3.732050808	2.143593539	-29.85640646	5.732050809	-2.267949192
210	0.577350269	1.333333333	-2.666666667	-0.577350269	1.732050808	1.539600718	-4.618802154	2.886751345	**-1.732050808**
180	0	1	-2	-1	1	2	-2	2.00	-2.00
150	-0.57735027	1.333333333	-2.666666667	-1.732050808	0.577350269	4.618802154	-1.539600718	**1.732050808**	-2.886751345
135	-1	2	-4	-2.414213562	0.414213562	9.656854249	-1.656854249	1.8284271247	-3.8284271247
120	-1.73205081	4	-8	-3.732050808	0.267949192	29.85640646	-2.143593539	2.267949192	-5.732050808
100	-5.67128182	33.16343748	-66.32687496	-11.4300523	0.087488664	758.1196498	-5.802849646	5.8462591466	-17.18882278704
80	5.67128182	33.16343748	-66.32687496	-0.087488664	11.4300523	5.802849646	-758.1196498	17.18882279	-5.846259147
70	2.747477419	8.54863217	-17.09726434	-0.176326981	5.67128182	3.014709	-96.96340442	8.595086218	-3.100131380471
60	1.732050808	4	-8	-0.267949192	3.732050808	2.143593539	-29.85640646	5.732050809	-2.267949192
50	1.191753593	2.420276625	-4.840553251	-0.363970234	2.747477419	1.761817301	-13.29931075	4.3032012473	-1.9196940611
45	1	2	-4	-0.414213562	2.414213562	1.656854249	-9.656854249	3.8284271247	-1.828427125
40	0.839099631	1.704088191	-3.408176382	-0.466307658	2.144506921	1.589258747	-7.308857838	3.44991420939	-1.771714948
35	0.700207538	1.490290597	-2.980581193	-0.520567051	1.920982127	1.551592361	-5.7256432	3.141756715	-1.741341639
30	0.577350269	1.333333333	-2.666666667	-0.577350269	1.732050808	1.539600718	-4.618802154	2.886751345	-1.732050808

After ascertaining these values for each of the *3θ Cubic Curve Sets* expressed in *Figure 32*, it now becomes possible to chart associated *S and T Curves* within their respective *real root regions*. This is depicted below in *Figure 34.* For each curve, S represents the lower portion, and T pertains to the upper portion. *S and T Curves* are joined, or connected, at respective left-most and right-most portions of each curve, respectively. Accordingly, *real root regions* are different for each *S and T Curve* represented.

Because of the great difference in relative sizes between *real root regions* mapped out by these sixteen curves, they are illustrated in groups of four in *Figure 35, Figure 36*, and *Figure 37*.

Lastly, *Table 31* represents the basis for such *Figure 34* thru *Figure 37* plots by charting *RST Curves* with respect to 'z'. For each of these *3θ Cubic Curves*, it indicates the spans over which the *S and T Curves* remain real and where they become imaginary. Therein, respective R, S, and T values are determined as follows:

$$R = \frac{z_f}{\tan\theta'}$$

$$S = \frac{z_1}{\tan\theta'}$$

$$T = \frac{z_2}{\tan\theta'}$$

Table 31 occupies a total of forty-eight pages, three pages for each of the sixteen *3θ Cubic Curves*, in order to properly account for the level of detail afforded in the accompanying plots.

Figure 34. S and T Curves for respective *3θ Cubic Curve* Sets.

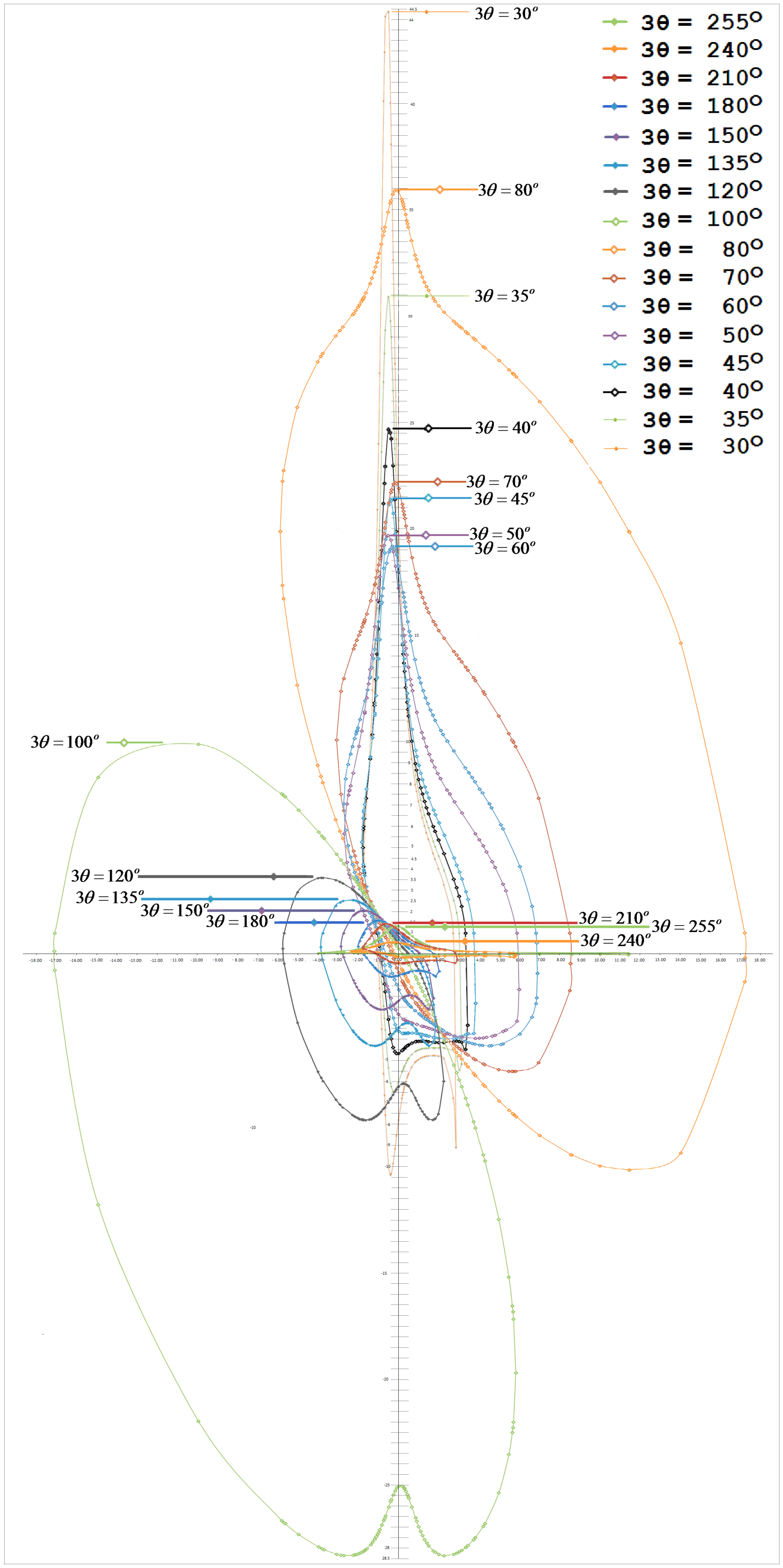


Figure 35. First Four S and T Curves Ranging from 255° thru 180°.

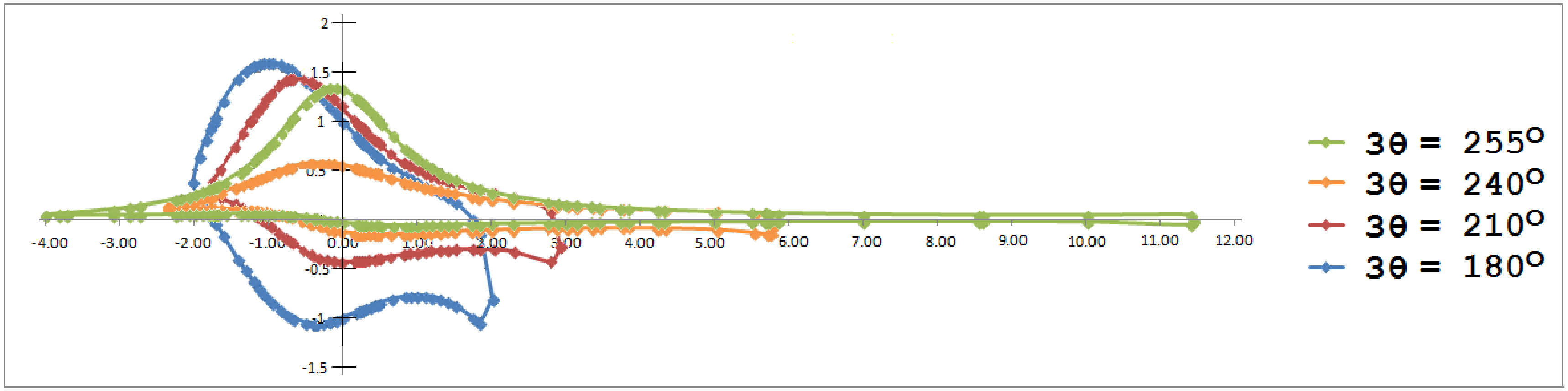


Figure 36. Second Four S and T Curves Ranging from 150° thru 100°.

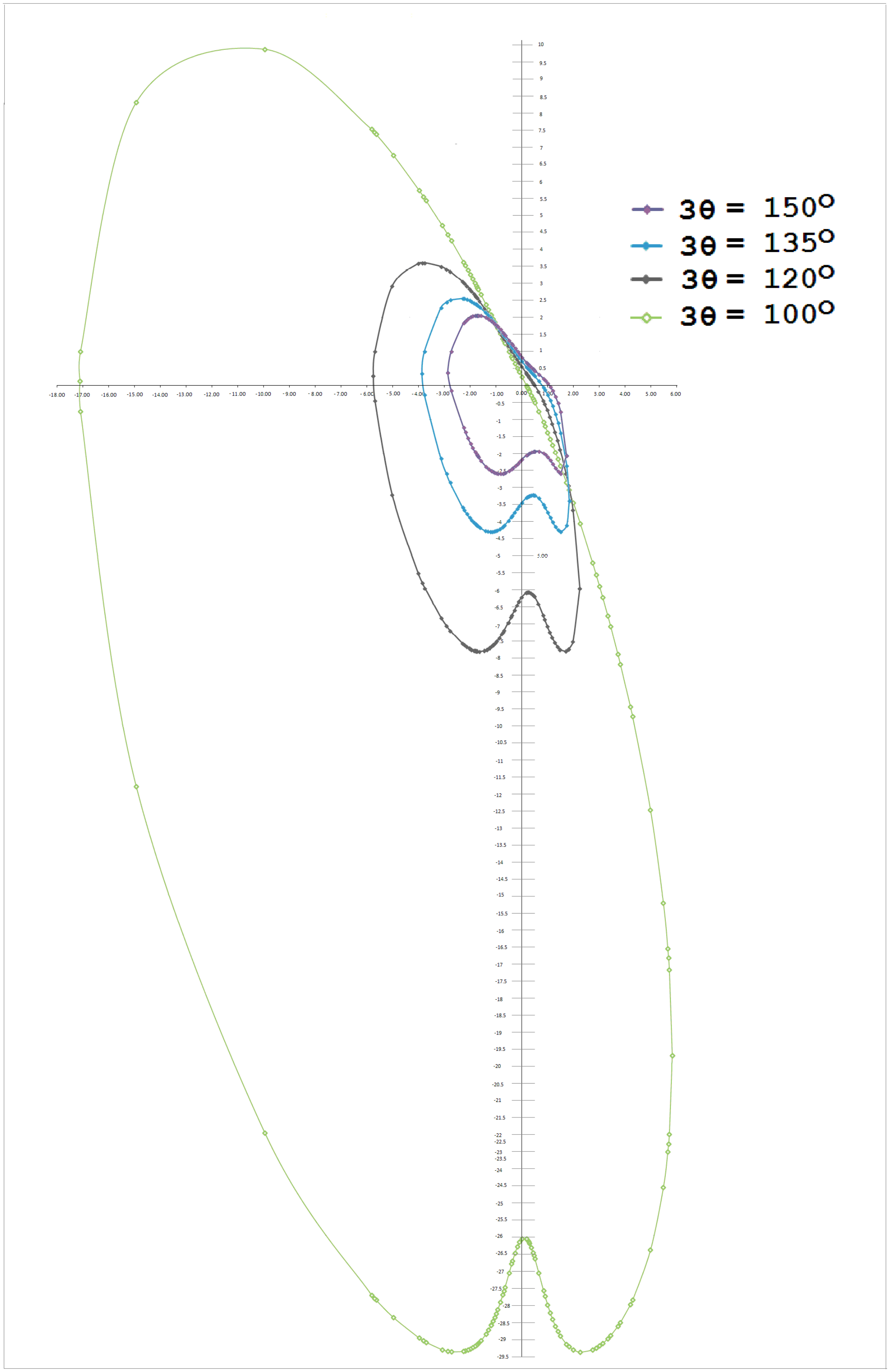
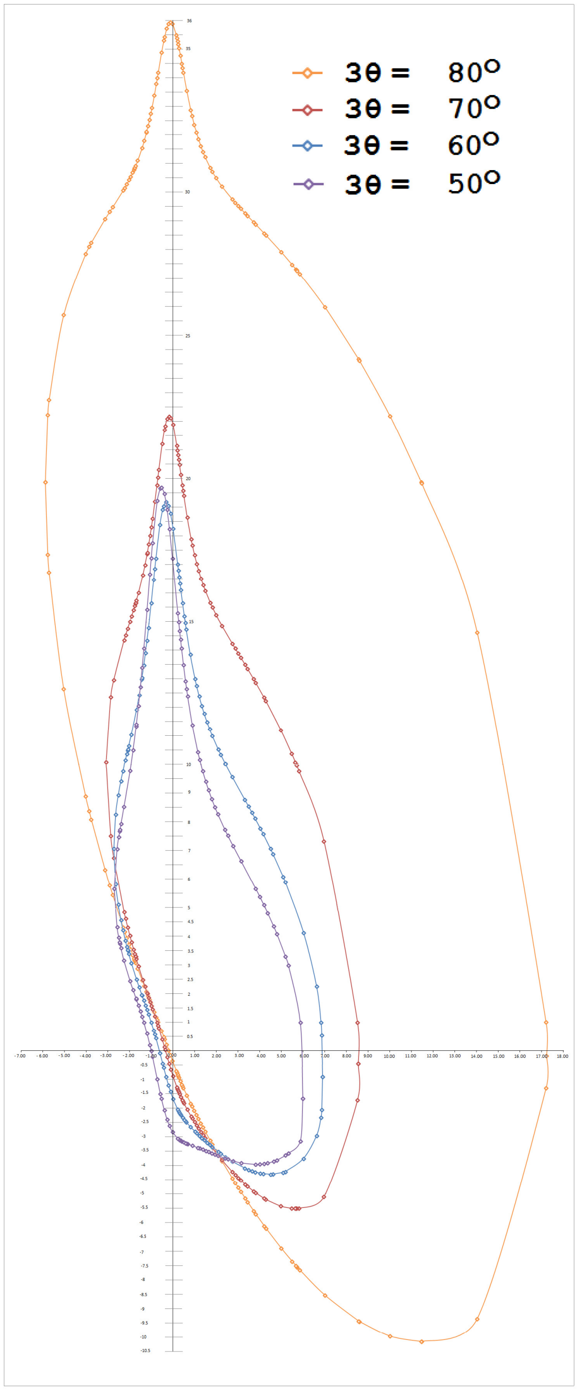


Figure 37. Third and Fourth S and T Curve Groups Ranging from 80° thru 30°.



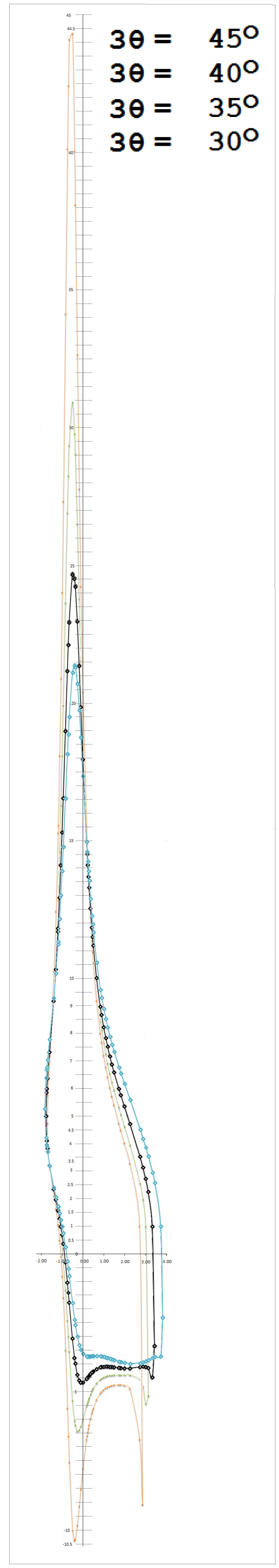


Table 31. Plot of RST Functions.

FOR 3θ= 255°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z=z_f$	$z^3-3\zeta z^2-3z+\zeta=y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta'=\zeta-y/4$
17.18882279	1722.733773	IMAG.	IMAG.	90.134197	30.044732	0.578392	29.718308	IMAG.	IMAG.	-426.951392
17.16933693	1713.035988	IMAG.	IMAG.	90.134964	30.044988	0.578398	29.684313	IMAG.	IMAG.	-424.526946
14	511.286176	IMAG.	IMAG.	90.461720	30.153907	0.580937	24.098982	IMAG.	IMAG.	-124.089493
11.45945741782	**3.930670**	-0.131652	-0.131652	250.012765	83.337588	8.561061	1.338556	-0.015378	-0.015378	2.749383
11.4300523	0.000000	0.466308	-0.700208	**255**	**85**	**11.430053**	1.000000	0.040797	-0.061260	**3.732051**
10	-145.883191	4.512054	-3.315901	268.575127	89.525042	120.630653	0.082898	0.037404	-0.027488	40.202849
8.595086218	-214.208293	6.501281	-3.900214	268.999898	89.666633	171.867888	0.050010	0.037827	-0.022693	57.284124
8.555546781	-215.220006	6.548594	-3.907988	269.004294	89.668098	172.626605	0.049561	0.037935	-0.022638	57.537052
7	-222.879418	8.162315	-3.966162	269.036358	89.678786	178.370666	0.039244	0.045760	-0.022236	59.451905
5.8462591466	-196.659370	9.111732	-3.761838	268.916969	89.638990	158.707483	0.036837	0.057412	-0.023703	52.896893
5.732050809	-192.994845	9.196152	-3.732051	268.897886	89.632629	155.959385	0.036753	0.058965	-0.023930	51.980762
5.70	-191.937941	9.219551	-3.723399	268.892257	89.630752	155.166795	0.036735	0.059417	-0.023996	51.716536
5.671281820	-190.980625	9.240409	-3.715539	268.887108	89.629036	154.448887	0.036719	0.059828	-0.024057	51.477207
5.50	-185.076560	9.362704	-3.666552	268.854264	89.618088	150.021348	0.036661	0.062409	-0.024440	50.001191
5.00	-166.171760	9.699503	-3.503351	268.734700	89.578233	135.844603	0.036807	0.071401	-0.025789	45.274991
4.3032012473	-136.818059	10.120301	-3.227350	268.490045	89.496682	113.833123	0.037803	0.088905	-0.028352	37.936565
4.219331772	-133.132612	10.167222	-3.190402	268.452478	89.484159	111.069621	0.037988	0.091539	-0.028724	37.015204
3.8284271247	-115.740906	10.375473	-3.007747	268.246628	89.415543	98.029036	0.039054	0.105841	-0.030682	32.667277
3.732050808	-111.425626	10.424181	-2.960080	268.186785	89.395595	94.793504	0.039370	0.109967	-0.031227	31.588457
3.44991420939	-98.812707	10.560788	-2.814549	267.985872	89.328624	85.336933	0.040427	0.123754	-0.032982	28.435228
3.340232616	-93.938354	10.611482	-2.755562	267.895772	89.298591	81.682566	0.040893	0.129911	-0.033735	27.216639
3.141756715	-85.195222	10.699774	-2.645378	267.712211	89.237404	75.128066	0.041819	0.142420	-0.035212	25.030856
3.017830135	-79.803864	10.752646	-2.574323	267.582159	89.194053	71.086566	0.042453	0.151261	-0.036214	23.683017
2.886751345	-74.173212	10.806671	-2.497270	267.429566	89.143189	66.865945	0.043172	0.161617	-0.037347	22.275354
2.747477419	-68.286321	10.861925	-2.413250	267.247994	89.082665	62.453595	0.043992	0.173920	-0.038641	20.803631
2.267949192	-48.994845	11.035016	-2.106813	266.419372	88.806457	47.997852	0.047251	0.229906	-0.043894	15.980762
2.00	-39.052559	11.119933	-1.923780	265.762101	88.587367	40.551345	0.049320	0.274219	-0.047441	13.495191
1.8284271247	-33.070902	11.169753	-1.802028	265.236270	88.412090	36.073271	0.050686	0.309641	-0.049955	11.999776
1.732050808	-29.856406	11.196152	-1.732051	264.896091	88.298697	33.667686	0.051445	0.332549	-0.051445	11.196152
1.50	-22.584292	11.254953	-1.558801	263.913485	87.971162	28.228877	0.053137	0.398704	-0.055220	9.378124

FOR 3θ= 255°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.40	-19.668408	11.278183	-1.482031	263.404845	87.801615	26.049876	0.053743	0.432946	-0.056892	8.649153
1.30	-16.892447	11.300120	-1.403967	262.835241	87.611747	23.976771	0.054219	0.471294	-0.058555	7.955163
1.191753595	-14.052224	11.322387	-1.317989	262.141449	87.380483	21.857408	0.054524	0.518011	-0.060299	7.245107
1.10	-11.784294	11.340043	-1.243890	261.483656	87.161219	20.166712	0.054545	0.562315	-0.061680	6.678124
1.00	-9.464102	11.357991	-1.161839	260.687174	86.895725	18.438992	0.054233	0.615977	-0.063010	6.098076
0.90	-7.307833	11.374570	-1.078418	259.802232	86.600744	16.835608	0.053458	0.675626	-0.064056	5.559009
0.839099631	-6.077527	11.383986	-1.026934	259.218575	86.406192	15.922001	0.052701	0.714985	-0.064498	5.251432
0.657710346	-2.799830	11.408922	-0.870480	257.285191	85.761730	13.494006	0.048741	0.845481	-0.064509	4.432008
0.502219976	-0.471885	11.426502	-0.732569	255.439804	85.146601	11.777041	0.042644	0.970235	-0.062203	3.850022
0.466307658	0.000000	11.430052	-0.700208	**255**	**85**	**11.430052**	0.040797	1.000000	-0.061260	**3.732051**
0.431357893	0.434976	11.433321	-0.668527	254.570975	84.856992	11.110582	0.038824	1.029048	-0.060170	3.623307
0.363970234	1.205154	11.439100	-0.606918	253.749631	84.583210	10.545910	0.034513	1.084695	-0.057550	3.430762
0.299380347	1.857247	11.443984	-0.547211	252.984737	84.328246	10.068932	0.029733	1.136564	-0.054347	3.267739
0.267949192	2.143594	11.446125	-0.517922	252.626341	84.208780	9.859845	0.027176	1.160883	-0.052528	3.196152
0.237004353	2.405451	11.448083	-0.488935	252.285551	84.095184	9.668851	0.024512	1.184017	-0.050568	3.130688
0.20648339	2.644052	11.449865	-0.460196	251.963569	83.987856	9.495005	0.021747	1.205883	-0.048467	3.071038
0.17632698	2.860450	11.451481	-0.431655	251.661609	83.887203	9.337498	0.018884	1.226397	-0.046228	3.016938
0.00	3.732051	**11.457979**	**-0.261826**	250.339940	83.446647	8.704812	0.000000	1.316281	-0.030078	2.799038
-0.10	3.919089	11.459371	-0.163219	250.032129	83.344043	8.569438	-0.011669	1.337237	-0.019047	2.752278
-0.17632698	3.907448	11.459285	-0.086805	250.051557	83.350519	8.577859	-0.020556	1.335914	-0.010120	2.755189
-0.267949192	3.712813	11.457835	0.006266	250.371083	83.457028	8.718744	-0.030733	1.314161	0.000719	2.803848
-0.363970234	3.292542	11.454704	0.105419	251.028529	83.676176	9.023486	-0.040336	1.269432	0.011683	2.908915
-0.40	3.076666	11.453094	0.143058	251.350000	83.783333	9.180284	-0.043572	1.247575	0.015583	2.962884
-0.502218876	2.288100	11.447206	0.251165	252.439863	84.146621	9.754419	-0.051486	1.173541	0.025749	3.160026
-0.657710346	0.577404	11.434391	0.419472	254.425245	84.808415	11.006059	-0.059759	1.038918	0.038113	3.587700
-0.700207538	0.000000	11.430052	0.466308	**255**	**85**	**11.430052**	-0.061260	1.000000	0.040797	**3.732051**
-0.744472416	-0.652494	11.425142	0.515483	255.601529	85.200510	11.909953	-0.062508	0.959294	0.043282	3.895174
-0.839099631	-2.224529	11.413277	0.621975	256.873277	85.624426	13.068994	-0.064205	0.873310	0.047592	4.288183
-0.943451341	-4.243062	11.397969	0.741634	258.214565	86.071522	14.561864	-0.064789	0.782727	0.050930	4.792816
-1.00	-5.464102	11.388670	0.807483	258.902195	86.300732	15.466881	-0.064654	0.736326	0.052207	5.098076
-1.059938076	-6.857469	11.378021	0.878070	259.595978	86.531993	16.501058	-0.064235	0.689533	0.053213	5.446418

FOR 3θ= 255°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.12369091	-8.452912	11.365778	0.954065	260.291922	86.763974	17.686769	-0.063533	0.642615	0.053942	5.845279
-1.191753595	-10.286942	11.351639	1.036267	260.986006	86.995335	19.051459	-0.062554	0.595841	0.054393	6.303786
-1.20	-10.518409	11.349849	1.046303	261.066670	87.022223	19.223801	-0.062423	0.590406	0.054427	6.361653
-1.30	-13.486447	11.326802	1.169350	261.986987	87.328996	21.435482	-0.060647	0.528414	0.054552	7.103663
-1.40	-16.756408	11.301190	1.294962	262.804801	87.601600	23.875216	-0.058638	0.473344	0.054239	7.921153
-1.60	-24.226099	11.241787	1.554365	264.166905	88.055635	29.456291	-0.054318	0.381643	0.052769	9.788576
-1.70	-28.437830	11.207722	1.688430	264.730058	88.243353	32.606334	-0.052137	0.343728	0.051782	10.841508
-1.732050808	-29.856406	11.196152	1.732051	264.896091	88.298697	33.667686	-0.051445	0.332549	0.051445	11.196152
-1.741341639	-30.273908	11.192738	1.744756	264.942987	88.314329	33.980084	-0.051246	0.329391	0.051346	11.300528
-1.771714948	-31.658601	11.181383	1.786484	265.092550	88.364183	35.016279	-0.050597	0.319320	0.051019	11.646701
-1.828427125	-34.325736	11.159377	1.865202	265.357097	88.452366	37.012519	-0.049400	0.301503	0.050394	12.313485
-1.9196940611	-38.843716	11.121688	1.994159	265.745702	88.581901	40.394968	-0.047523	0.275324	0.049367	13.442980
-2.00	-43.052559	11.086094	2.110058	266.053509	88.684503	43.546819	-0.045928	0.254579	0.048455	14.495191
-2.10	-48.603981	11.038406	2.257746	266.397401	88.799134	47.705046	-0.044021	0.231389	0.047327	15.883046
-2.20	-54.505327	10.986746	2.409406	266.702889	88.900963	52.126312	-0.042205	0.210772	0.046222	17.358383
-2.267949192	-58.717968	10.949231	2.514871	266.891106	88.963702	55.282870	-0.041024	0.198058	0.045491	18.411543
-2.747477419	-93.280850	10.618237	3.325393	267.882998	89.294333	81.189636	-0.033840	0.130783	0.040958	27.052263
-2.886751345	-104.965226	10.495175	3.587729	268.089152	89.363051	89.949720	-0.032093	0.116678	0.039886	29.973357
-3.100131380471	-124.366488	10.274733	4.021550	268.355141	89.451714	104.496538	-0.029667	0.098326	0.038485	34.823673
-3.732050808	-192.994845	9.196152	5.732051	268.897886	89.632629	155.959385	-0.023930	0.058965	0.036753	51.980762
-3.8284271247	-204.995740	8.895760	6.128820	268.958013	89.652671	164.959124	-0.023208	0.053927	0.037154	54.980986
-3.9953558031	-226.781922	7.595754	7.595754	269.051913	89.683971	181.297303	-0.022038	0.041897	0.041897	60.427531
-5	-386.171760	IMAG.	IMAG.	269.428632	89.809544	300.833837	-0.016620	IMAG.	IMAG.	100.274991
-5.67128182	-521.768770	IMAG.	IMAG.	269.572983	89.857661	402.529355	-0.014089	IMAG.	IMAG.	134.174243
-5.732050808	-535.271723	IMAG.	IMAG.	269.583462	89.861154	412.656407	-0.013891	IMAG.	IMAG.	137.549981
-5.846259147	-561.217428	IMAG.	IMAG.	269.602220	89.867407	432.115395	-0.013529	IMAG.	IMAG.	144.036408
-10	-2085.883191	IMAG.	IMAG.	269.890907	89.963636	1575.610238	-0.006347	IMAG.	IMAG.	525.202849
-15	-5845.402244	IMAG.	IMAG.	269.960892	89.986964	4395.248442	-0.003413	IMAG.	IMAG.	1465.082612
-17.16933693	-8306.512788	IMAG.	IMAG.	269.972459	89.990820	6241.081170	-0.002751	IMAG.	IMAG.	2080.360248
-17.18882278704	-8331.202180	IMAG.	IMAG.	269.972540	89.990847	6259.598213	-0.002746	IMAG.	IMAG.	2086.532596

FOR 3θ= 240°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	3493.467546	IMAG.	IMAG.	90.065734	30.021911	0.577860	29.745638	IMAG.	IMAG.	-871.634836
17.16933693	3479.752772	IMAG.	IMAG.	90.065993	30.021998	0.577862	29.711814	IMAG.	IMAG.	-868.206142
14	1685.286176	IMAG.	IMAG.	90.136552	30.045517	0.578410	24.204285	IMAG.	IMAG.	-419.589493
11.45945741782	789.845656	IMAG.	IMAG.	90.292727	30.097576	0.579623	19.770529	IMAG.	IMAG.	-195.729363
11.4300523	781.876573	IMAG.	IMAG.	90.295737	30.098579	0.579647	19.719002	IMAG.	IMAG.	-193.737092
10	452.116809	IMAG.	IMAG.	90.514786	30.171595	0.581350	17.201330	IMAG.	IMAG.	-111.297151
8.595086218	227.044750	IMAG.	IMAG.	91.041075	30.347025	0.585454	14.681053	IMAG.	IMAG.	-55.029137
8.555546781	221.964278	IMAG.	IMAG.	91.065666	30.355222	0.585646	14.608723	IMAG.	IMAG.	-53.759019
7	69.120582	IMAG.	IMAG.	93.679999	31.226666	0.606258	11.546243	IMAG.	IMAG.	-15.548095
5.8462591466	6.413106	IMAG.	IMAG.	187.337842	62.445947	1.916565	3.050383	IMAG.	IMAG.	0.128774
5.732050809	2.143594	-0.267949	-0.267949	230.103909	76.701303	4.230727	1.354862	-0.063334	-0.063334	1.196152
5.70	1.002059	0.185723	-0.689570	235.981631	78.660544	4.986637	1.143055	0.037244	-0.138284	1.481536
5.671281820	0.000000	0.363970	-0.839100	**240**	**80**	**5.671282**	1.000000	0.064178	-0.147956	**1.732051**
5.50	-5.576560	1.010797	-1.314644	252.261664	84.087221	9.655738	0.569610	0.104684	-0.136152	3.126191
5.00	-18.171760	2.095670	-1.899518	260.945325	86.981775	18.965707	0.263634	0.110498	-0.100155	6.274991
4.3032012473	-27.712813	3.100131	-2.207180	263.413224	87.804408	26.083050	0.164981	0.118856	-0.084621	8.660254
4.219331772	-28.316049	3.201356	-2.224536	263.524997	87.841666	26.533738	0.159018	0.120652	-0.083838	8.811063
3.8284271247	-29.799781	3.634101	-2.266375	263.784484	87.928161	27.642500	0.138498	0.131468	-0.081989	9.181996
3.732050808	-29.856406	3.732051	-2.267949	263.793977	87.931326	27.684822	0.134805	0.134805	-0.081920	9.196152
3.44991420939	-29.401258	4.001491	-2.255253	263.716844	87.905615	27.344660	0.126164	0.146335	-0.082475	9.082365
3.340232616	-28.995430	4.099760	-2.243840	263.646440	87.882147	27.041384	0.123523	0.151611	-0.082978	8.980908
3.141756715	-27.971410	4.269041	-2.214645	263.461607	87.820536	26.276245	0.119566	0.162468	-0.084283	8.724903
3.017830135	-27.160071	4.369417	-2.191095	263.307381	87.769127	25.670141	0.117562	0.170214	-0.085356	8.522069
2.886751345	-26.173212	4.471325	-2.161924	263.109743	87.703248	24.933072	0.115780	0.179333	-0.086709	8.275354
2.747477419	-24.994528	4.574962	-2.126287	262.857916	87.619305	24.052979	0.114226	0.190204	-0.088400	7.980683
2.267949192	-20.133284	4.896974	-1.968770	261.591904	87.197301	20.426767	0.111028	0.239733	-0.096382	6.765372
2.00	-17.052559	5.054397	-1.858244	260.530224	86.843408	18.132784	0.110297	0.278744	-0.102480	5.995191
1.8284271247	-15.012027	5.146959	-1.779233	259.667685	86.555895	16.615856	0.110041	0.309762	-0.107080	5.485058
1.732050808	-13.856406	5.196152	-1.732051	259.106605	86.368868	15.757912	0.109916	0.329749	-0.109916	5.196152
1.50	-11.084292	5.306344	-1.610191	257.479609	85.826536	13.704304	0.109455	0.387203	-0.117495	4.503124
1.40	-9.908408	5.350221	-1.554069	256.635578	85.545193	12.835635	0.109071	0.416826	-0.121075	4.209153

FOR 3θ= 240°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-8.752447	5.391911	-1.495759	255.689528	85.229843	11.983533	0.108482	0.449943	-0.124818	3.920163
1.191753595	-7.530564	5.434554	-1.430155	254.535961	84.845320	11.085289	0.107508	0.490249	-0.129014	3.614692
1.10	-6.524294	5.468659	-1.372506	253.440545	84.480182	10.347879	0.106302	0.528481	-0.132636	3.363124
1.00	-5.464102	5.503671	-1.307519	252.110909	84.036970	9.573784	0.104452	0.574869	-0.136573	3.098076
0.90	-4.447833	5.536404	-1.240252	250.627493	83.542498	8.835146	0.101866	0.626634	-0.140377	2.844009
0.839099631	-3.852998	5.555208	-1.198155	249.644331	83.214777	8.404688	0.099837	0.660965	-0.142558	2.695300
0.657710346	-2.204333	5.606038	-1.067595	246.346848	82.115616	7.221068	0.091082	0.776345	-0.147845	2.283134
0.502219976	-0.958536	5.643273	-0.949340	243.106767	81.035589	6.339233	0.079224	0.890214	-0.149756	1.971685
0.466307658	-0.695343	5.651017	-0.921173	242.314444	80.771481	6.154774	0.075764	0.918152	-0.149668	1.905887
0.431357893	-0.448606	5.658240	-0.893446	241.531687	80.510562	5.982539	0.072103	0.945792	-0.149342	1.844202
0.363970234	0.000000	5.671282	-0.839100	**240**	**80**	**5.671282**	0.064178	1.000000	-0.147956	**1.732051**
0.299380347	0.395019	5.682670	-0.785898	238.522578	79.507526	5.399475	0.055446	1.052449	-0.145551	1.633296
0.267949192	0.574374	5.687811	-0.759608	237.807876	79.269292	5.276847	0.050778	1.077881	-0.143951	1.588457
0.237004353	0.742477	5.692614	-0.733466	237.111270	79.037090	5.162394	0.045910	1.102708	-0.142079	1.546432
0.20648339	0.899864	5.697097	-0.707428	236.434426	78.811475	5.055683	0.040842	1.126870	-0.139927	1.507085
0.17632698	1.046997	5.701275	-0.681449	235.779102	78.593034	4.956337	0.035576	1.150300	-0.137491	1.470301
0.00	1.732051	**5.720575**	**-0.524423**	232.410911	77.470304	4.499670	0.000000	1.271332	-0.116547	1.299038
-0.10	1.979089	5.727474	-0.431322	231.053973	77.017991	4.337690	-0.023054	1.320397	-0.099436	1.237278
-0.17632698	2.093995	5.730672	-0.358193	230.394406	76.798135	4.262898	-0.041363	1.344314	-0.084026	1.208552
-0.267949192	2.143594	5.732051	-0.267949	230.103909	76.701303	4.230727	-0.063334	1.354862	-0.063334	1.196152
-0.363970234	2.087388	5.730489	-0.170366	230.432836	76.810945	4.267188	-0.085295	1.342919	-0.039925	1.210204
-0.40	2.036666	5.729078	-0.132925	230.725797	76.908599	4.300167	-0.093020	1.332292	-0.030912	1.222884
-0.502218876	1.801443	5.722516	-0.024145	232.037941	77.345980	4.454013	-0.112756	1.284800	-0.005421	1.281690
-0.657710346	1.172901	5.704841	0.149022	235.200264	78.400088	4.871658	-0.135007	1.171027	0.030590	1.438826
-0.700207538	0.941744	5.698287	0.198073	236.250162	78.750054	5.027364	-0.139279	1.133454	0.039399	1.496615
-0.744472416	0.672941	5.690629	0.249995	237.402646	79.134215	5.209679	-0.142902	1.092319	0.047987	1.563816
-0.839099631	0.000000	5.671282	0.363970	**240**	**80**	**5.671282**	-0.147956	1.000000	0.064178	**1.732051**
-0.943451341	-0.902459	5.644926	0.494677	242.941488	80.980496	6.299871	-0.149757	0.896038	0.078522	1.957666
-1.00	-1.464102	5.628279	0.567874	244.516265	81.505422	6.695490	-0.149354	0.840607	0.084814	2.098076
-1.059938076	-2.116657	5.608690	0.647400	246.143063	82.047688	7.158597	-0.148065	0.783490	0.090437	2.261215
-1.12369091	-2.876824	5.585526	0.734318	247.806801	82.602267	7.701959	-0.145897	0.725208	0.095342	2.451257

FOR 3θ= 240°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-3.765282	5.557959	0.829947	249.491211	83.163737	8.341346	-0.142873	0.666314	0.099498	2.673371
-1.20	-3.878409	5.554410	0.841743	249.688281	83.229427	8.423045	-0.142466	0.659430	0.099933	2.701653
-1.30	-5.346447	5.507501	0.988651	251.950512	83.983504	9.488086	-0.137014	0.580465	0.104199	3.068663
-1.40	-6.996408	5.452767	1.143385	253.972697	84.657566	10.693559	-0.130920	0.509911	0.106923	3.481153
-1.60	-10.866099	5.314604	1.481549	257.331012	85.777004	13.542989	-0.118142	0.392425	0.109396	4.448576
-1.70	-13.097830	5.227337	1.668815	258.704392	86.234797	15.195272	-0.111877	0.344011	0.109825	5.006508
-1.732050808	-13.856406	5.196152	1.732051	259.106605	86.368868	15.757912	-0.109916	0.329749	0.109916	5.196152
-1.741341639	-14.080284	5.186786	1.750708	259.219957	86.406652	15.924048	-0.109353	0.325720	0.109941	5.252122
-1.771714948	-14.824758	5.155077	1.812791	259.580675	86.526892	16.476764	-0.107528	0.312869	0.110021	5.438240
-1.828427125	-16.266861	5.090975	1.933604	260.215552	86.738517	17.548423	-0.104193	0.290110	0.110187	5.798766
-1.9196940611	-18.732364	4.971631	2.144216	261.139969	87.046656	19.383125	-0.099039	0.256493	0.110623	6.415142
-2.00	-21.052559	4.844610	2.351542	261.864383	87.288128	21.111977	-0.094733	0.229472	0.111384	6.995191
-2.10	-24.143981	4.641323	2.654829	262.664515	87.554838	23.418080	-0.089674	0.198194	0.113367	7.768046
-2.20	-27.465327	4.333869	3.062284	263.366249	87.788750	25.898166	-0.084948	0.167343	0.118243	8.598383
-2.267949192	-29.856406	3.732051	3.732051	263.793977	87.931326	27.684822	-0.081920	0.134805	0.134805	9.196152
-2.747477419	-49.989057	IMAG.	IMAG.	265.980008	88.660003	42.750334	-0.064268	IMAG.	IMAG.	14.229315
-2.886751345	-56.965227	IMAG.	IMAG.	266.417716	88.805905	47.975664	-0.060171	IMAG.	IMAG.	15.973357
-3.100131380471	-68.701600	IMAG.	IMAG.	266.972493	88.990831	56.769331	-0.054609	IMAG.	IMAG.	18.907451
-3.732050808	-111.425626	IMAG.	IMAG.	268.064313	89.354771	88.795405	-0.042030	IMAG.	IMAG.	29.588457
-3.8284271247	-119.054615	IMAG.	IMAG.	268.181449	89.393816	94.515329	-0.040506	IMAG.	IMAG.	31.495705
-3.9953558031	-133.004714	IMAG.	IMAG.	268.362639	89.454213	104.975092	-0.038060	IMAG.	IMAG.	34.983229
-5	-238.171760	IMAG.	IMAG.	269.065023	89.688341	183.839478	-0.027198	IMAG.	IMAG.	61.274991
-5.67128182	-330.788146	IMAG.	IMAG.	269.321406	89.773802	253.297789	-0.022390	IMAG.	IMAG.	84.429087
-5.732050808	-340.133284	IMAG.	IMAG.	269.339676	89.779892	260.306360	-0.022020	IMAG.	IMAG.	86.765372
-5.846259147	-358.144952	IMAG.	IMAG.	269.372252	89.790751	273.814605	-0.021351	IMAG.	IMAG.	91.268289
-10	-1487.883191	IMAG.	IMAG.	269.846681	89.948894	1121.110925	-0.008920	IMAG.	IMAG.	373.702849
-15	-4497.402244	IMAG.	IMAG.	269.949119	89.983040	3378.248625	-0.004440	IMAG.	IMAG.	1126.082612
-17.16933693	-6539.796004	IMAG.	IMAG.	269.964993	89.988331	4910.043699	-0.003497	IMAG.	IMAG.	1636.681052
-17.18882278704	-6560.468407	IMAG.	IMAG.	269.965103	89.988368	4925.547999	-0.003490	IMAG.	IMAG.	1641.849153

FOR 3θ= 210°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	4515.801166	IMAG.	IMAG.	90.050777	30.016926	0.577744	29.751614	IMAG.	IMAG.	-1128.372941
17.16933693	4499.767183	IMAG.	IMAG.	90.050958	30.016986	0.577746	29.717814	IMAG.	IMAG.	-1124.364445
14	2363.095392	IMAG.	IMAG.	90.097079	30.032360	0.578104	24.217114	IMAG.	IMAG.	-590.196498
11.45945741782	1243.593885	IMAG.	IMAG.	90.184633	30.061544	0.578783	19.799217	IMAG.	IMAG.	-310.321121
11.4300523	1233.293223	IMAG.	IMAG.	90.186178	30.062059	0.578795	19.748003	IMAG.	IMAG.	-307.745956
10	797.372270	IMAG.	IMAG.	90.288255	30.096085	0.579588	17.253622	IMAG.	IMAG.	-198.765717
8.595086218	481.802313	IMAG.	IMAG.	90.477959	30.159320	0.581064	14.791984	IMAG.	IMAG.	-119.873228
8.555546781	474.372743	IMAG.	IMAG.	90.485481	30.161827	0.581122	14.722454	IMAG.	IMAG.	-118.015835
7	237.706861	IMAG.	IMAG.	90.973507	30.324502	0.584927	11.967313	IMAG.	IMAG.	-58.849365
5.8462591466	123.657055	IMAG.	IMAG.	91.887965	30.629322	0.592089	9.873948	IMAG.	IMAG.	-30.336913
5.732050809	114.806824	IMAG.	IMAG.	92.036372	30.678791	0.593256	9.662019	IMAG.	IMAG.	-28.124356
5.70	112.396020	IMAG.	IMAG.	92.080928	30.693643	0.593606	9.602321	IMAG.	IMAG.	-27.521655
5.671281820	110.262715	IMAG.	IMAG.	92.122013	30.707338	0.593930	9.548741	IMAG.	IMAG.	-26.988329
5.50	98.057813	IMAG.	IMAG.	92.392206	30.797402	0.596058	9.227287	IMAG.	IMAG.	-23.937103
5.00	67.276080	IMAG.	IMAG.	93.523255	31.174418	0.605011	8.264306	IMAG.	IMAG.	-16.241670
4.3032012473	35.279130	IMAG.	IMAG.	96.917511	32.305837	0.632316	6.805456	IMAG.	IMAG.	-8.242432
4.219331772	32.199823	IMAG.	IMAG.	97.622159	32.540720	0.638070	6.612649	IMAG.	IMAG.	-7.472605
3.8284271247	19.818351	IMAG.	IMAG.	102.868639	34.289546	0.681886	5.614464	IMAG.	IMAG.	-4.377237
3.732050808	17.237604	IMAG.	IMAG.	105.000000	35.000000	0.700208	5.329921	IMAG.	IMAG.	-3.732051
3.44991420939	10.673460	IMAG.	IMAG.	115.558839	38.519613	0.795995	4.334090	IMAG.	IMAG.	-2.091015
3.340232616	8.499384	IMAG.	IMAG.	122.870758	40.956919	0.867968	3.848338	IMAG.	IMAG.	-1.547496
3.141756715	5.066773	IMAG.	IMAG.	145.419835	48.473278	1.129233	2.782205	IMAG.	IMAG.	-0.689343
3.017830135	3.233836	IMAG.	IMAG.	166.986912	55.662304	1.463875	2.061535	IMAG.	IMAG.	-0.231109
2.886751345	1.539601	-0.577350	-0.577350	190.893395	63.631132	2.017238	1.431041	-0.286208	-0.286208	0.192450
2.747477419	0.000000	0.176327	-1.191754	**210**	**70**	**2.747477**	1.000000	0.064178	-0.433763	**0.577350**
2.267949192	-3.470054	1.094551	-1.630449	235.312628	78.437543	4.887878	0.463995	0.223932	-0.333570	1.444864
2.00	-4.350853	1.441478	-1.709427	239.011912	79.670637	5.486660	0.364521	0.262724	-0.311561	1.665064
1.8284271247	-4.585731	1.632914	-1.729290	239.881147	79.960382	5.648440	0.323705	0.289091	-0.306154	1.723783
1.732050808	-4.618802	1.732051	-1.732051	240.000000	80.000000	5.671282	0.305407	0.305407	-0.305407	1.732051
1.50	-4.444764	1.949475	-1.717424	239.364831	79.788277	5.551249	0.270209	0.351178	-0.309376	1.688541
1.40	-4.273469	2.034831	-1.702781	238.715558	79.571853	5.433536	0.257659	0.374495	-0.313383	1.645718

FOR 3θ= 210°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-4.052816	2.115587	-1.683536	237.841945	79.280648	5.282570	0.246092	0.400484	-0.318696	1.590554
1.191753595	-3.765282	2.198070	-1.657773	236.636273	78.878758	5.087059	0.234272	0.432091	-0.325880	1.518671
1.10	-3.487431	2.264134	-1.632083	235.393081	78.464360	4.899555	0.224510	0.462110	-0.333108	1.449208
1.00	-3.154701	2.332247	-1.600196	233.793977	77.931326	4.677058	0.213810	0.498657	-0.342137	1.366025
0.90	-2.796611	2.396410	-1.564359	231.925197	77.308399	4.440385	0.202685	0.539685	-0.352303	1.276503
0.839099631	-2.568665	2.433588	-1.540637	230.648347	76.882782	4.291402	0.195530	0.567085	-0.359005	1.219517
0.657710346	-1.860522	2.535959	-1.461619	226.191502	75.397167	3.838281	0.171355	0.660702	-0.380800	1.042481
0.502219976	-1.239504	2.613857	-1.384026	221.580278	73.860093	3.455546	0.145337	0.756424	-0.400523	0.887226
0.466307658	-1.096799	2.630558	-1.364815	220.416060	73.472020	3.369899	0.138374	0.780604	-0.405002	0.851550
0.431357893	-0.958743	2.646346	-1.345653	219.250059	73.083353	3.287953	0.131193	0.804861	-0.409268	0.817036
0.363970234	-0.695796	2.675485	-1.307404	216.917511	72.305837	3.134517	0.116117	0.853556	-0.417099	0.751299
0.299380347	-0.449199	2.701795	-1.269124	214.592089	71.530696	2.994015	0.099993	0.902399	-0.423887	0.689650
0.267949192	-0.331615	2.714020	-1.249919	213.434949	71.144983	2.928261	0.091505	0.926837	-0.426847	0.660254
0.237004353	-0.217641	2.725685	-1.230638	212.283082	70.761027	2.865332	0.082714	0.951263	-0.429492	0.631761
0.20648339	-0.107143	2.736826	-1.211259	211.137687	70.379229	2.805108	0.073610	0.975658	-0.431805	0.604136
0.17632698	0.000000	2.747477	-1.191754	**210**	**70**	**2.747477**	0.064178	1.000000	-0.433763	**0.577350**
0.00	0.577350	2.802517	-1.070466	203.413224	67.804408	2.450964	0.000000	1.143435	-0.436753	0.433013
-0.10	0.859030	2.828066	-0.996015	199.930281	66.643427	2.315678	-0.043184	1.221269	-0.430118	0.362593
-0.17632698	1.046997	2.844686	-0.936308	197.515744	65.838581	2.229114	-0.079102	1.276151	-0.420036	0.315601
-0.267949192	1.237604	2.861210	-0.861210	195.000000	65.000000	2.144507	-0.124947	1.334204	-0.401589	0.267949
-0.363970234	1.391592	2.874328	-0.778307	192.922955	64.307652	2.078557	-0.175107	1.382848	-0.374446	0.229452
-0.40	1.436222	2.878093	-0.746042	192.314191	64.104730	2.059852	-0.194189	1.397233	-0.362183	0.218295
-0.502218876	1.520471	2.885156	-0.650886	191.157387	63.719129	2.025048	-0.248003	1.424734	-0.321418	0.197233
-0.657710346	1.516711	2.884842	-0.495081	191.209211	63.736404	2.026587	-0.324541	1.423498	-0.244293	0.198172
-0.700207538	1.485459	2.882228	-0.449969	191.639298	63.879766	2.039431	-0.343335	1.413251	-0.220635	0.205985
-0.744472416	1.438181	2.878258	-0.401735	192.287402	64.095801	2.059035	-0.361564	1.397868	-0.195108	0.217805
-0.839099631	1.284333	2.865212	-0.294062	194.373700	64.791233	2.124264	-0.395007	1.348802	-0.138430	0.256267
-0.943451341	1.026239	2.842866	-0.167364	197.785753	65.928584	2.238523	-0.421461	1.269974	-0.074765	0.320791
-1.00	0.845299	2.826838	-0.094788	200.103909	66.701303	2.322119	-0.430641	1.217353	-0.040819	0.366025
-1.059938076	0.620452	2.806478	-0.014490	202.891280	67.630427	2.429844	-0.436217	1.155004	-0.005963	0.422237
-1.12369091	0.342531	2.780588	0.075154	206.184149	68.728050	2.568582	-0.437475	1.082538	0.029259	0.491717

FOR 3θ= 210°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	0.000000	2.747477	0.176327	**210**	**70**	**2.747477**	-0.433763	1.000000	0.064178	**0.577350**
-1.20	-0.044803	2.743041	0.189010	210.478981	70.159660	2.771483	-0.432981	0.989738	0.068198	0.588551
-1.30	-0.646816	2.680786	0.351265	216.466408	72.155469	3.106339	-0.418499	0.863005	0.113080	0.739054
-1.40	-1.361469	2.599256	0.532795	222.543150	74.181050	3.529469	-0.396660	0.736444	0.150956	0.917718
-1.60	-3.152700	2.332628	0.999423	233.783975	77.927992	4.675727	-0.342193	0.498880	0.213747	1.365525
-1.70	-4.241277	2.048075	1.383976	238.590766	79.530255	5.411463	-0.314148	0.378470	0.255749	1.637669
-1.732050808	-4.618802	1.732051	1.732051	240.000000	80.000000	5.671282	-0.305407	0.305407	0.305407	1.732051
-1.741341639	-4.730891	IMAG.	IMAG.	240.396570	80.132190	5.748809	-0.302905	IMAG.	IMAG.	1.760073
-1.771714948	-5.105734	IMAG.	IMAG.	241.655924	80.551975	6.009246	-0.294831	IMAG.	IMAG.	1.853784
-1.828427125	-5.840565	IMAG.	IMAG.	243.858215	81.286072	6.524421	-0.280244	IMAG.	IMAG.	2.037492
-1.9196940611	-7.121070	IMAG.	IMAG.	247.015433	82.338478	7.433754	-0.258240	IMAG.	IMAG.	2.357618
-2.00	-8.350853	IMAG.	IMAG.	249.432624	83.144208	8.317358	-0.240461	IMAG.	IMAG.	2.665064
-2.10	-10.021994	IMAG.	IMAG.	252.028217	84.009406	9.529412	-0.220370	IMAG.	IMAG.	3.082849
-2.20	-11.853776	IMAG.	IMAG.	254.229126	84.743042	10.868436	-0.202421	IMAG.	IMAG.	3.540794
-2.267949192	-13.193176	IMAG.	IMAG.	255.532011	85.177337	11.852457	-0.191348	IMAG.	IMAG.	3.875644
-2.747477419	-24.994528	IMAG.	IMAG.	261.665510	87.221837	20.607452	-0.133324	IMAG.	IMAG.	6.825982
-2.886751345	-29.252414	IMAG.	IMAG.	262.777102	87.592367	23.783549	-0.121376	IMAG.	IMAG.	7.890454
-3.100131380471	-36.563463	IMAG.	IMAG.	264.124968	88.041656	29.245864	-0.106002	IMAG.	IMAG.	9.718216
-3.732050808	-64.331615	IMAG.	IMAG.	266.565051	88.855017	50.034066	-0.074590	IMAG.	IMAG.	16.660254
-3.8284271247	-69.436483	IMAG.	IMAG.	266.808931	88.936310	53.858931	-0.071082	IMAG.	IMAG.	17.936471
-3.9953558031	-78.862418	IMAG.	IMAG.	267.178850	89.059617	60.922640	-0.065581	IMAG.	IMAG.	20.292955
-5	-152.723920	IMAG.	IMAG.	268.522045	89.507348	116.297921	-0.042993	IMAG.	IMAG.	38.758330
-5.67128182	-220.525430	IMAG.	IMAG.	268.971622	89.657207	167.142078	-0.033931	IMAG.	IMAG.	55.708708
-5.732050808	-227.470054	IMAG.	IMAG.	269.002696	89.667565	172.350064	-0.033258	IMAG.	IMAG.	57.444864
-5.846259147	-240.901003	IMAG.	IMAG.	269.057760	89.685920	182.422422	-0.032048	IMAG.	IMAG.	60.802601
-10	-1142.627730	IMAG.	IMAG.	269.799830	89.933277	858.705954	-0.011645	IMAG.	IMAG.	286.234283
-15	-3719.134081	IMAG.	IMAG.	269.938416	89.979472	2791.083567	-0.005374	IMAG.	IMAG.	930.360871
-17.16933693	-5519.781593	IMAG.	IMAG.	269.958497	89.986166	4141.568890	-0.004146	IMAG.	IMAG.	1380.522749
-17.18882278704	-5538.134787	IMAG.	IMAG.	269.958635	89.986212	4155.333783	-0.004137	IMAG.	IMAG.	1385.111047

FOR 3θ= 180°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	5026.967977	IMAG.	IMAG.	90.045591	30.015197	0.577704	29.753686	IMAG.	IMAG.	-1256.741994
17.16933693	5009.774388	IMAG.	IMAG.	90.045747	30.015249	0.577705	29.719894	IMAG.	IMAG.	-1252.443597
14	2702.000000	IMAG.	IMAG.	90.084820	30.028273	0.578008	24.221101	IMAG.	IMAG.	-675.500000
11.45945741782	1470.467999	IMAG.	IMAG.	90.155857	30.051952	0.578560	19.806865	IMAG.	IMAG.	-367.617000
11.4300523	1459.001548	IMAG.	IMAG.	90.157082	30.052361	0.578569	19.755715	IMAG.	IMAG.	-364.750387
10	970.000000	IMAG.	IMAG.	90.236270	30.078757	0.579184	17.265656	IMAG.	IMAG.	-242.500000
8.595086218	609.181094	IMAG.	IMAG.	90.376210	30.125403	0.580272	14.812162	IMAG.	IMAG.	-152.295274
8.555546781	600.576975	IMAG.	IMAG.	90.381599	30.127200	0.580314	14.742957	IMAG.	IMAG.	-150.144244
7	322.000000	IMAG.	IMAG.	90.711712	30.237237	0.582884	12.009244	IMAG.	IMAG.	-80.500000
5.8462591466	182.279029	IMAG.	IMAG.	91.257119	30.419040	0.587143	9.957125	IMAG.	IMAG.	-45.569757
5.732050809	171.138439	IMAG.	IMAG.	91.338924	30.446308	0.587783	9.751977	IMAG.	IMAG.	-42.784610
5.70	168.093000	IMAG.	IMAG.	91.363173	30.454391	0.587973	9.694318	IMAG.	IMAG.	-42.023250
5.671281820	165.394073	IMAG.	IMAG.	91.385409	30.461803	0.588147	9.642620	IMAG.	IMAG.	-41.348518
5.50	149.875000	IMAG.	IMAG.	91.528799	30.509600	0.589271	9.333571	IMAG.	IMAG.	-37.468750
5.00	110.000000	IMAG.	IMAG.	92.082565	30.694188	0.593619	8.422906	IMAG.	IMAG.	-27.500000
4.3032012473	66.775102	IMAG.	IMAG.	93.428068	31.142689	0.604255	7.121496	IMAG.	IMAG.	-16.693775
4.219331772	62.457758	IMAG.	IMAG.	93.664405	31.221468	0.606134	6.961057	IMAG.	IMAG.	-15.614440
3.8284271247	44.627417	IMAG.	IMAG.	95.121792	31.707264	0.617788	6.196994	IMAG.	IMAG.	-11.156854
3.732050808	40.784610	IMAG.	IMAG.	95.601439	31.867146	0.621650	6.003461	IMAG.	IMAG.	-10.196152
3.44991420939	30.710819	IMAG.	IMAG.	97.420843	32.473614	0.636423	5.420788	IMAG.	IMAG.	-7.677705
3.340232616	27.246792	IMAG.	IMAG.	98.351722	32.783907	0.644059	5.186225	IMAG.	IMAG.	-6.811698
3.141756715	21.585864	IMAG.	IMAG.	100.498195	33.499398	0.661870	4.746785	IMAG.	IMAG.	-5.396466
3.017830135	18.430790	IMAG.	IMAG.	102.244903	34.081634	0.676584	4.460396	IMAG.	IMAG.	-4.607698
2.886751345	15.396007	IMAG.	IMAG.	104.563891	34.854630	0.696433	4.145052	IMAG.	IMAG.	-3.849002
2.747477419	12.497264	IMAG.	IMAG.	107.748312	35.916104	0.724308	3.793246	IMAG.	IMAG.	-3.124316
2.267949192	4.861561	IMAG.	IMAG.	129.446897	43.148966	0.937388	2.419436	IMAG.	IMAG.	-1.215390
2.00	2.000000	-1.000000	-1.000000	153.434949	51.144983	1.241306	1.611206	-0.805603	-0.805603	-0.500000
1.8284271247	0.627417	-0.212330	-1.616097	171.085548	57.028516	1.541544	1.186101	-0.137738	-1.048363	-0.156854
1.732050808	0.000000	0.000000	-1.732051	**180**	**60**	**1.732051**	1.000000	0.000000	-1.000000	**0.000000**
1.50	-1.125000	0.395644	-1.895644	195.708638	65.236213	2.167796	0.691947	0.182510	-0.874457	0.281250
1.40	-1.456000	0.536932	-1.936932	200.001506	66.667169	2.318316	0.603887	0.231604	-0.835491	0.364000

FOR 3θ= 180°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-1.703000	0.666245	-1.966245	203.061880	67.687293	2.436713	0.533506	0.273419	-0.806925	0.425750
1.191753595	-1.882641	0.795091	-1.986845	205.204502	68.401501	2.525905	0.471813	0.314775	-0.786587	0.470660
1.10	-1.969000	0.896548	-1.996548	206.208717	68.736239	2.569669	0.428071	0.348896	-0.776967	0.492250
1.00	-2.000000	1.000000	-2.000000	206.565051	68.855017	2.585515	0.386770	0.386770	-0.773540	0.500000
0.90	-1.971000	1.096771	-1.996771	206.231773	68.743924	2.570689	0.350101	0.426645	-0.776745	0.492750
0.839099631	-1.926499	1.152689	-1.991788	205.716614	68.572205	2.548060	0.329309	0.452379	-0.781688	0.481625
0.657710346	-1.688617	1.306860	-1.964570	202.887244	67.629081	2.429681	0.270698	0.537873	-0.808571	0.422154
0.502219976	-1.379988	1.425443	-1.927663	199.034275	66.344758	2.282907	0.219991	0.624398	-0.844309	0.344997
0.466307658	-1.297528	1.451161	-1.917469	197.972126	65.990709	2.245057	0.207704	0.646381	-0.854085	0.324382
0.431357893	-1.213811	1.475607	-1.906965	196.880566	65.626855	2.207237	0.195429	0.668531	-0.863960	0.303453
0.363970234	-1.043694	1.521143	-1.885113	194.623767	64.874589	2.132309	0.170693	0.713378	-0.884071	0.260923
0.299380347	-0.871308	1.562846	-1.862226	192.288609	64.096203	2.059071	0.145396	0.759005	-0.904401	0.217827
0.267949192	-0.784610	1.582461	-1.850411	191.097805	63.699268	2.023281	0.132433	0.782126	-0.914559	0.196152
0.237004353	-0.697700	1.601344	-1.838349	189.894280	63.298093	1.988114	0.119211	0.805459	-0.924670	0.174425
0.20648339	-0.610647	1.619554	-1.826037	188.679854	62.893285	1.953607	0.105693	0.829007	-0.934700	0.152662
0.17632698	-0.523499	1.637143	-1.813470	187.456189	62.485396	1.919787	0.091847	0.852773	-0.944620	0.130875
0.00	0.000000	**1.732051**	**-1.732051**	**180**	**60**	**1.732051**	0.000000	1.000000	-1.000000	**0.000000**
-0.10	0.299000	1.779884	-1.679884	175.725091	58.575030	1.636659	-0.061100	1.087511	-1.026411	-0.074750
-0.17632698	0.523499	1.813470	-1.637143	172.543811	57.514604	1.570569	-0.112270	1.154658	-1.042388	-0.130875
-0.267949192	0.784610	1.850411	-1.582461	168.902195	56.300732	1.499478	-0.178695	1.234036	-1.055341	-0.196152
-0.363970234	1.043694	1.885113	-1.521143	165.376233	55.125411	1.434822	-0.253669	1.31383	-1.060161	-0.260923
-0.40	1.136000	1.897056	-1.497056	164.145454	54.715151	1.413143	-0.283057	1.342438	-1.059381	-0.284000
-0.502218876	1.379985	1.927663	-1.425444	160.965757	53.655252	1.359109	-0.369521	1.418329	-1.048808	-0.344996
-0.657710346	1.688617	1.964570	-1.306860	157.112756	52.370919	1.297164	-0.507037	1.514512	-1.007474	-0.422154
-0.700207538	1.757317	1.972535	-1.272327	156.282706	52.094235	1.284290	-0.545210	1.535895	-0.990685	-0.439329
-0.744472416	1.820801	1.979818	-1.235346	155.524954	51.841651	1.272676	-0.584966	1.555634	-0.970668	-0.455200
-0.839099631	1.926499	1.991788	-1.152689	154.283386	51.427795	1.253925	-0.669178	1.588442	-0.919264	-0.481625
-0.943451341	1.990588	1.998953	-1.055502	153.542909	51.180970	1.242904	-0.759070	1.608293	-0.849223	-0.497647
-1.00	2.000000	2.000000	-1.000000	153.434949	51.144983	1.241306	-0.805603	1.611206	-0.805603	-0.500000
-1.059938076	1.989007	1.998778	-0.938839	153.561058	51.187019	1.243172	-0.852608	1.607804	-0.755197	-0.497252
-1.12369091	1.952209	1.994671	-0.870980	153.985203	51.328401	1.249473	-0.899332	1.596410	-0.697078	-0.488052

FOR 3θ= 180°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	1.882641	1.986845	-0.795091	154.795498	51.598499	1.261618	-0.944623	1.574839	-0.630216	-0.470660
-1.20	1.872000	1.985641	-0.785641	154.920406	51.640135	1.263503	-0.949740	1.571536	-0.621796	-0.468000
-1.30	1.703000	1.966245	-0.666245	156.938120	52.312707	1.294442	-1.004294	1.51899	-0.514696	-0.425750
-1.40	1.456000	1.936932	-0.536932	159.998494	53.332831	1.343209	-1.042280	1.442019	-0.399738	-0.364000
-1.60	0.704000	1.839230	-0.239230	170.018171	56.672724	1.520776	-1.052094	1.209402	-0.157308	-0.176000
-1.70	0.187000	1.762414	-0.062414	177.323371	59.107790	1.671394	-1.017115	1.054458	-0.037343	-0.046750
-1.732050808	0.000000	1.732051	0.000000	**180**	**60**	**1.732051**	-1.000000	1.000000	0.000000	**0.000000**
-1.741341639	-0.056194	1.722608	0.018734	180.804871	60.268290	1.750934	-0.994521	0.983822	0.010699	0.014049
-1.771714948	-0.246222	1.689455	0.082260	183.522427	61.174142	1.817050	-0.975050	0.929779	0.045271	0.061556
-1.828427125	-0.627417	1.616097	0.212330	188.914452	62.971484	1.960198	-0.932777	0.824456	0.108321	0.156854
-1.9196940611	-1.315423	1.445729	0.473965	198.203746	66.067915	2.253221	-0.851978	0.641628	0.210350	0.328856
-2.00	-2.000000	1.000000	1.000000	206.565051	68.855017	2.585515	-0.773540	0.386770	0.386770	0.500000
-2.10	-2.961000	IMAG.	IMAG.	216.510696	72.170232	3.109085	-0.675440	IMAG.	IMAG.	0.740250
-2.20	-4.048000	IMAG.	IMAG.	225.341720	75.113907	3.761951	-0.584803	IMAG.	IMAG.	1.012000
-2.267949192	-4.861561	IMAG.	IMAG.	230.553103	76.851034	4.280669	-0.529812	IMAG.	IMAG.	1.215390
-2.747477419	-12.497264	IMAG.	IMAG.	252.251688	84.083896	9.650271	-0.284705	IMAG.	IMAG.	3.124316
-2.886751345	-15.396007	IMAG.	IMAG.	255.436109	85.145370	11.774038	-0.245179	IMAG.	IMAG.	3.849002
-3.100131380471	-20.494394	IMAG.	IMAG.	258.956114	86.318705	15.542604	-0.199460	IMAG.	IMAG.	5.123598
-3.732050808	-40.784610	IMAG.	IMAG.	264.398561	88.132854	30.675420	-0.121663	IMAG.	IMAG.	10.196152
-3.8284271247	-44.627417	IMAG.	IMAG.	264.878208	88.292736	33.550069	-0.114111	IMAG.	IMAG.	11.156854
-3.9953558031	-51.791270	IMAG.	IMAG.	265.583637	88.527879	38.911998	-0.102677	IMAG.	IMAG.	12.947817
-5	-110.000000	IMAG.	IMAG.	267.917435	89.305812	82.532312	-0.060582	IMAG.	IMAG.	27.500000
-5.67128182	-165.394073	IMAG.	IMAG.	268.614591	89.538197	124.067049	-0.045711	IMAG.	IMAG.	41.348518
-5.732050808	-171.138439	IMAG.	IMAG.	268.661076	89.553692	128.374602	-0.044651	IMAG.	IMAG.	42.784610
-5.846259147	-182.279029	IMAG.	IMAG.	268.742881	89.580960	136.728775	-0.042758	IMAG.	IMAG.	45.569757
-10	-970.000000	IMAG.	IMAG.	269.763730	89.921243	727.503666	-0.013746	IMAG.	IMAG.	242.500000
-15	-3330.000000	IMAG.	IMAG.	269.931176	89.977059	2497.501068	-0.006006	IMAG.	IMAG.	832.500000
-17.16933693	-5009.774388	IMAG.	IMAG.	269.954253	89.984751	3757.331501	-0.004570	IMAG.	IMAG.	1252.443597
-17.18882278704	-5026.967977	IMAG.	IMAG.	269.954409	89.984803	3770.226690	-0.004559	IMAG.	IMAG.	1256.741994

FOR 3θ=150°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	5538.134787	IMAG.	IMAG.	90.041365	30.013788	0.577671	29.755375	IMAG.	IMAG.	-1385.111047
17.16933693	5519.781593	IMAG.	IMAG.	90.041503	30.013834	0.577672	29.721588	IMAG.	IMAG.	-1380.522749
14	3040.904608	IMAG.	IMAG.	90.075310	30.025103	0.577935	24.224194	IMAG.	IMAG.	-760.803502
11.45945741782	1697.342114	IMAG.	IMAG.	90.134841	30.044947	0.578397	19.812453	IMAG.	IMAG.	-424.912879
11.4300523	1684.709873	IMAG.	IMAG.	90.135851	30.045284	0.578405	19.761346	IMAG.	IMAG.	-421.754819
10	1142.627730	IMAG.	IMAG.	90.200170	30.066723	0.578904	17.274020	IMAG.	IMAG.	-286.234283
8.595086218	736.559876	IMAG.	IMAG.	90.310178	30.103393	0.579759	14.825279	IMAG.	IMAG.	-184.717319
8.555546781	726.781207	IMAG.	IMAG.	90.314338	30.104779	0.579791	14.756256	IMAG.	IMAG.	-182.272652
7	406.293139	IMAG.	IMAG.	90.560877	30.186959	0.581709	12.033503	IMAG.	IMAG.	-102.150635
5.8462591466	240.901003	IMAG.	IMAG.	90.942240	30.314080	0.584683	9.999032	IMAG.	IMAG.	-60.802601
5.732050809	227.470054	IMAG.	IMAG.	90.997304	30.332435	0.585112	9.796494	IMAG.	IMAG.	-57.444864
5.70	223.789980	IMAG.	IMAG.	91.013533	30.337844	0.585239	9.739607	IMAG.	IMAG.	-56.524845
5.671281820	220.525430	IMAG.	IMAG.	91.028378	30.342793	0.585355	9.688617	IMAG.	IMAG.	-55.708708
5.50	201.692187	IMAG.	IMAG.	91.123294	30.374431	0.586097	9.384115	IMAG.	IMAG.	-51.000397
5.00	152.723920	IMAG.	IMAG.	91.477955	30.492652	0.588872	8.490806	IMAG.	IMAG.	-38.758330
4.3032012473	98.271073	IMAG.	IMAG.	92.277404	30.759135	0.595153	7.230407	IMAG.	IMAG.	-25.145119
4.219331772	92.715694	IMAG.	IMAG.	92.410394	30.803465	0.596202	7.077022	IMAG.	IMAG.	-23.756274
3.8284271247	69.436483	IMAG.	IMAG.	93.191069	31.063690	0.602375	6.355559	IMAG.	IMAG.	-17.936471
3.732050808	64.331615	IMAG.	IMAG.	93.434949	31.144983	0.604310	6.175724	IMAG.	IMAG.	-16.660254
3.44991420939	50.748178	IMAG.	IMAG.	94.311361	31.437120	0.611292	5.643642	IMAG.	IMAG.	-13.264395
3.340232616	45.994199	IMAG.	IMAG.	94.733838	31.577946	0.614674	5.434156	IMAG.	IMAG.	-12.075900
3.141756715	38.104956	IMAG.	IMAG.	95.652425	31.884142	0.622061	5.050559	IMAG.	IMAG.	-10.103589
3.017830135	33.627744	IMAG.	IMAG.	96.351190	32.117063	0.627714	4.807652	IMAG.	IMAG.	-8.984286
2.886751345	29.252414	IMAG.	IMAG.	97.222898	32.407633	0.634806	4.547453	IMAG.	IMAG.	-7.890454
2.747477419	24.994528	IMAG.	IMAG.	98.334490	32.778163	0.643917	4.266821	IMAG.	IMAG.	-6.825982
2.267949192	13.193176	IMAG.	IMAG.	104.467989	34.822663	0.695605	3.260399	IMAG.	IMAG.	-3.875644
2.00	8.350853	IMAG.	IMAG.	110.567376	36.855792	0.749615	2.668035	IMAG.	IMAG.	-2.665064
1.8284271247	5.840565	IMAG.	IMAG.	116.141785	38.713928	0.801550	2.281113	IMAG.	IMAG.	-2.037492
1.732050808	4.618802	-1.732051	-1.732051	120.000000	40.000000	0.839100	2.064178	-2.064178	-2.064178	-1.732051
1.50	2.194764	-0.742262	-2.489788	131.607220	43.869073	0.961282	1.560416	-0.772158	-2.590070	-1.126041
1.40	1.361469	-0.532795	-2.599256	137.456850	45.818950	1.029003	1.360540	-0.517778	-2.525993	-0.917718

FOR 3θ=150°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	0.646816	-0.351265	-2.680786	143.533592	47.844531	1.104570	1.176928	-0.318010	-2.426994	-0.739054
1.191753595	0.000000	-0.176327	-2.747477	**150**	**50**	**1.191754**	1.000000	-0.147956	-2.305407	**-0.577350**
1.10	-0.450569	-0.041299	-2.790752	155.075327	51.691776	1.265846	0.868984	-0.032626	-2.204653	-0.464708
1.00	-0.845299	0.094788	-2.826838	159.896091	53.298697	1.341539	0.745412	0.070656	-2.107160	-0.366025
0.90	-1.145389	0.221205	-2.853256	163.774846	54.591615	1.406701	0.639795	0.157251	-2.028332	-0.291003
0.839099631	-1.284333	0.294062	-2.865212	165.626300	55.208767	1.439281	0.582999	0.204311	-1.990724	-0.256267
0.657710346	-1.516711	0.495081	-2.884842	168.790789	56.263596	1.497375	0.439242	0.330632	-1.926600	-0.198172
0.502219976	-1.520471	0.650885	-2.885156	168.842621	56.280874	1.498353	0.335181	0.434400	-1.925552	-0.197232
0.466307658	-1.498256	0.684941	-2.883299	168.536648	56.178883	1.492592	0.312415	0.458893	-1.931740	-0.202786
0.431357893	-1.468879	0.717429	-2.880838	168.133063	56.044354	1.485039	0.290469	0.483104	-1.939907	-0.210130
0.363970234	-1.391592	0.778307	-2.874328	167.077045	55.692348	1.465525	0.248355	0.531078	-1.961296	-0.229452
0.299380347	-1.293417	0.834557	-2.865988	165.748473	55.249491	1.441467	0.207691	0.578964	-1.988244	-0.253996
0.267949192	-1.237604	0.861210	-2.861210	165.000000	55.000000	1.428148	0.187620	0.603026	-2.003441	-0.267949
0.237004353	-1.177759	0.887001	-2.856056	164.203239	54.734413	1.414151	0.167595	0.627232	-2.019626	-0.282910
0.20648339	-1.114150	0.912010	-2.850544	163.363186	54.454395	1.399591	0.147531	0.651626	-2.036699	-0.298813
0.17632698	-1.046997	0.936308	-2.844686	162.484256	54.161419	1.384568	0.127352	0.676245	-2.054565	-0.315601
0.00	-0.577350	**1.070466**	**-2.802517**	156.586776	52.195592	1.288987	0.000000	0.830471	-2.174200	-0.433013
-0.10	-0.261030	1.140784	-2.772835	152.883340	50.961113	1.233185	-0.081091	0.925071	-2.248515	-0.512093
-0.17632698	0.000000	1.191754	-2.747477	**150**	**50**	**1.191754**	-0.147956	1.000000	-2.305407	**-0.577350**
-0.267949192	0.331615	1.249919	-2.714020	146.565051	48.855017	1.144506	-0.234118	1.092103	-2.371346	-0.660254
-0.363970234	0.695796	1.307404	-2.675485	143.082489	47.694163	1.098761	-0.331255	1.189890	-2.435002	-0.751299
-0.40	0.835778	1.328068	-2.660119	141.822216	47.274072	1.082706	-0.369445	1.226619	-2.456917	-0.786295
-0.502218876	1.239499	1.384025	-2.613857	138.419758	46.139919	1.040604	-0.482623	1.330021	-2.511866	-0.887225
-0.657710346	1.860522	1.461619	-2.535959	133.808498	44.602833	0.986231	-0.666892	1.482024	-2.571363	-1.042481
-0.700207538	2.029175	1.481246	-2.513090	132.674861	44.224954	0.973305	-0.719412	1.521872	-2.582016	-1.084644
-0.744472416	2.203422	1.500967	-2.488545	131.552599	43.850866	0.960671	-0.774950	1.562415	-2.590424	-1.128206
-0.839099631	2.568665	1.540637	-2.433588	129.351653	43.117218	0.936347	-0.896141	1.645369	-2.599023	-1.219517
-0.943451341	2.954936	1.580410	-2.369009	127.228648	42.409549	0.913431	-1.032865	1.7301900	-2.593528	-1.316084
-1.00	3.154701	1.600196	-2.332247	126.206023	42.068674	0.902577	-1.107939	1.772919	-2.583987	-1.366025
-1.059938076	3.357562	1.619789	-2.291902	125.216183	41.738728	0.892181	-1.188031	1.815539	-2.568876	-1.416741
-1.12369091	3.561887	1.639048	-2.247408	124.265880	41.421960	0.882300	-1.273593	1.857699	-2.547215	-1.467822

FOR 3θ=150°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	3.765282	1.657773	-2.198070	123.363727	41.121242	0.873009	-1.365111	1.898919	-2.517810	-1.518671
-1.20	3.788803	1.659911	-2.191962	123.262104	41.087368	0.871967	-1.376198	1.903639	-2.513811	-1.524551
-1.30	4.052816	1.683536	-2.115587	122.158055	40.719352	0.860723	-1.510357	1.955954	-2.457917	-1.590554
-1.40	4.273469	1.702781	-2.034831	121.284442	40.428147	0.851914	-1.643358	1.998707	-2.388540	-1.645718
-1.60	4.560700	1.727195	-1.859246	120.209380	40.069793	0.841178	-1.902095	2.053307	-2.210290	-1.717525
-1.70	4.615277	1.731757	-1.763808	120.012630	40.004210	0.839225	-2.025679	2.063520	-2.101711	-1.731169
-1.732050808	4.618802	1.732051	-1.732051	120.000000	40.000000	0.839100	-2.064178	2.064178	-2.064178	-1.732051
-1.741341639	4.618502	1.732026	-1.722735	120.001074	40.000358	0.839110	-2.075224	2.064122	-2.053050	-1.731976
-1.771714948	4.613290	1.731591	-1.691927	120.019751	40.006584	0.839295	-2.110955	2.063149	-2.015890	-1.730673
-1.828427125	4.585731	1.729290	-1.632914	120.118853	40.039618	0.840279	-2.175977	2.057997	-1.943301	-1.723783
-1.9196940611	4.490224	1.721269	-1.533626	120.466925	40.155642	0.843739	-2.275222	2.040048	-1.817654	-1.699906
-2.00	4.350853	1.709427	-1.441478	120.988088	40.329363	0.848943	-2.355870	2.013594	-1.697968	-1.665064
-2.10	4.099994	1.687688	-1.319739	121.967622	40.655874	0.858797	-2.445282	1.965177	-1.536730	-1.602349
-2.20	3.757776	1.657089	-1.189140	123.396275	41.132092	0.873342	-2.519058	1.897411	-1.361597	-1.516794
-2.267949192	3.470054	1.630449	-1.094551	124.687372	41.562457	0.886670	-2.557826	1.838845	-1.234451	-1.444864
-2.747477419	0.000000	1.191754	-0.176327	**150**	**50**	**1.191754**	-2.305407	1.000000	-0.147956	**-0.577350**
-2.886751345	-1.539601	0.577350	0.577350	169.106605	56.368868	1.503348	-1.920215	0.384043	0.384043	-0.192450
-3.100131380471	-4.425325	IMAG.	IMAG.	207.877987	69.292662	2.645399	-1.171896	IMAG.	IMAG.	0.528981
-3.732050808	-17.237604	IMAG.	IMAG.	255.000000	85.000000	11.430052	-0.326512	IMAG.	IMAG.	3.732051
-3.8284271247	-19.818351	IMAG.	IMAG.	257.131361	85.710454	13.332107	-0.287158	IMAG.	IMAG.	4.377237
-3.9953558031	-24.720122	IMAG.	IMAG.	259.880072	86.626691	16.965405	-0.235500	IMAG.	IMAG.	5.602680
-5	-67.276080	IMAG.	IMAG.	266.476745	88.825582	48.779684	-0.102502	IMAG.	IMAG.	16.241670
-5.67128182	-110.262715	IMAG.	IMAG.	267.877987	89.292662	80.997910	-0.070018	IMAG.	IMAG.	26.988329
-5.732050808	-114.806824	IMAG.	IMAG.	267.963628	89.321209	84.404662	-0.067912	IMAG.	IMAG.	28.124356
-5.846259147	-123.657055	IMAG.	IMAG.	268.112035	89.370678	91.040033	-0.064216	IMAG.	IMAG.	30.336913
-10	-797.372270	IMAG.	IMAG.	269.711745	89.903915	596.301623	-0.016770	IMAG.	IMAG.	198.765717
-15	-2940.865919	IMAG.	IMAG.	269.922008	89.974003	2203.918598	-0.006806	IMAG.	IMAG.	734.639129
-17.16933693	-4499.767183	IMAG.	IMAG.	269.949042	89.983014	3373.094127	-0.005090	IMAG.	IMAG.	1124.364445
-17.18882278704	-4515.801166	IMAG.	IMAG.	269.949223	89.983074	3385.119612	-0.005078	IMAG.	IMAG.	1128.372941

FOR 3θ= 135°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	5912.334863	IMAG.	IMAG.	90.038737	30.012912	0.577651	29.756425	IMAG.	IMAG.	-1479.083716
17.16933693	5893.132780	IMAG.	IMAG.	90.038863	30.012954	0.577652	29.722642	IMAG.	IMAG.	-1474.283195
14	3289.000000	IMAG.	IMAG.	90.069597	30.023199	0.577890	24.226053	IMAG.	IMAG.	-823.250000
11.45945741782	1863.425492	IMAG.	IMAG.	90.122727	30.040909	0.578303	19.815675	IMAG.	IMAG.	-466.856373
11.4300523	1849.939835	IMAG.	IMAG.	90.123619	30.041206	0.578310	19.764591	IMAG.	IMAG.	-463.484959
10	1269.000000	IMAG.	IMAG.	90.180033	30.060011	0.578748	17.278688	IMAG.	IMAG.	-318.250000
8.595086218	829.807616	IMAG.	IMAG.	90.274861	30.091620	0.579484	14.832301	IMAG.	IMAG.	-208.451904
8.555546781	819.169117	IMAG.	IMAG.	90.278413	30.092804	0.579512	14.763366	IMAG.	IMAG.	-205.792279
7	468.000000	IMAG.	IMAG.	90.485546	30.161849	0.581123	12.045646	IMAG.	IMAG.	-118.000000
5.8462591466	283.815267	IMAG.	IMAG.	90.796234	30.265411	0.583543	10.018553	IMAG.	IMAG.	-71.953817
5.732050809	268.707658	IMAG.	IMAG.	90.840338	30.280113	0.583887	9.817050	IMAG.	IMAG.	-68.176915
5.70	264.563000	IMAG.	IMAG.	90.853305	30.284435	0.583988	9.760467	IMAG.	IMAG.	-67.140750
5.671281820	260.884385	IMAG.	IMAG.	90.865154	30.288385	0.584081	9.709754	IMAG.	IMAG.	-66.221096
5.50	239.625000	IMAG.	IMAG.	90.940636	30.313545	0.584670	9.407016	IMAG.	IMAG.	-60.906250
5.00	184.000000	IMAG.	IMAG.	91.218875	30.406292	0.586844	8.520150	IMAG.	IMAG.	-47.000000
4.3032012473	121.327725	IMAG.	IMAG.	91.828050	30.609350	0.591619	7.273607	IMAG.	IMAG.	-31.331931
4.219331772	114.866040	IMAG.	IMAG.	91.927352	30.642451	0.592399	7.122451	IMAG.	IMAG.	-29.716510
3.8284271247	87.597980	IMAG.	IMAG.	92.500466	30.833489	0.596912	6.413720	IMAG.	IMAG.	-22.899495
3.732050808	81.569219	IMAG.	IMAG.	92.676388	30.892129	0.598301	6.237747	IMAG.	IMAG.	-21.392305
3.44991420939	65.416543	IMAG.	IMAG.	93.297916	31.099305	0.603222	5.719145	IMAG.	IMAG.	-17.354136
3.340232616	59.718253	IMAG.	IMAG.	93.592107	31.197369	0.605559	5.515951	IMAG.	IMAG.	-15.929563
3.141756715	50.197770	IMAG.	IMAG.	94.220992	31.406997	0.610570	5.145611	IMAG.	IMAG.	-13.549443
3.017830135	44.752686	IMAG.	IMAG.	94.690427	31.563476	0.614326	4.912427	IMAG.	IMAG.	-12.188172
2.886751345	39.396007	IMAG.	IMAG.	95.266322	31.755441	0.618950	4.663948	IMAG.	IMAG.	-10.849002
2.747477419	34.143161	IMAG.	IMAG.	95.986617	31.995539	0.624761	4.397645	IMAG.	IMAG.	-9.535790
2.267949192	19.292342	IMAG.	IMAG.	99.744370	33.248123	0.655582	3.459445	IMAG.	IMAG.	-5.823085
2.00	13.000000	IMAG.	IMAG.	103.240520	34.413507	0.685061	2.919450	IMAG.	IMAG.	-4.250000
1.8284271247	9.656854	-2.414214	-2.414214	106.324950	35.441650	0.711758	2.568890	-3.391904	-3.391904	-3.414214
1.732050808	8.000000	-1.732051	-3.000000	108.434949	36.144983	0.730416	2.371322	-2.371322	-4.107250	-3.000000
1.50	4.625000	-1.104356	-3.395644	114.880367	38.293456	0.789567	1.899776	-1.398686	-4.300641	-2.156250
1.40	3.424000	-0.903852	-3.496148	118.315481	39.438494	0.822535	1.702055	-1.098861	-4.250455	-1.856000

FOR 3θ= 135°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	2.367000	-0.724342	-3.575658	122.138655	40.712885	0.860527	1.510702	-0.841743	-4.155195	-1.591750
1.191753595	1.378189	-0.547600	-3.644154	126.639834	42.213278	0.907167	1.313709	-0.603637	-4.017071	-1.344547
1.10	0.661000	-0.409116	-3.690884	130.635696	43.545232	0.950466	1.157327	-0.430437	-3.883236	-1.165250
1.00	0.000000	-0.267949	-3.732051	**135**	**45**	**1.000000**	1.000000	-0.267949	-3.732051	**-1.000000**
0.90	-0.541000	-0.135475	-3.764525	139.148389	46.382796	1.049472	0.857574	-0.129089	-3.587064	-0.864750
0.839099631	-0.814234	-0.058560	-3.780540	141.464731	47.154910	1.078198	0.778242	-0.054312	-3.506349	-0.796441
0.657710346	-1.390868	0.155836	-3.813546	146.884274	48.961425	1.148805	0.572517	0.135650	-3.319575	-0.652283
0.502219976	-1.623313	0.324342	-3.826562	149.282413	49.760804	1.181700	0.424998	0.274471	-3.238185	-0.594172
0.466307658	-1.645199	0.361472	-3.827779	149.514670	49.838223	1.184943	0.393528	0.305054	-3.230350	-0.588700
0.431357893	-1.655602	0.397000	-3.828358	149.625456	49.875152	1.186493	0.363557	0.334599	-3.226615	-0.586099
0.363970234	-1.646271	0.463869	-3.827839	149.526072	49.842024	1.185102	0.307121	0.391417	-3.229965	-0.588432
0.299380347	-1.602422	0.526018	-3.825399	149.061762	49.687254	1.178628	0.254008	0.446297	-3.245637	-0.599394
0.267949192	-1.569219	0.555598	-3.823547	148.713153	49.571051	1.173794	0.228276	0.473335	-3.257426	-0.607695
0.237004353	-1.529187	0.584306	-3.821311	148.296241	49.432080	1.168043	0.202907	0.500244	-3.271550	-0.617703
0.20648339	-1.482740	0.612226	-3.818710	147.817182	49.272394	1.161475	0.177777	0.527111	-3.287811	-0.629315
0.17632698	-1.430225	0.639434	-3.815761	147.281538	49.093846	1.154181	0.152772	0.554015	-3.306033	-0.642444
0.00	-1.000000	**0.791288**	**-3.791288**	143.130102	47.710034	1.099372	0.000000	0.719763	-3.448593	-0.750000
-0.10	-0.671000	0.872176	-3.772176	140.231080	46.743693	1.062797	-0.094091	0.820642	-3.549291	-0.832250
-0.17632698	-0.383228	0.931488	-3.755162	137.880331	45.960110	1.034089	-0.170514	0.900782	-3.631373	-0.904193
-0.267949192	0.000000	1.000000	-3.732051	**135**	**45**	**1.000000**	-0.267949	1.000000	-3.732051	**-1.000000**
-0.363970234	0.441117	1.068740	-3.704770	132.008551	44.002850	0.965785	-0.376865	1.106603	-3.836020	-1.110279
-0.40	0.616000	1.093742	-3.693742	130.910603	43.636868	0.953515	-0.419501	1.147063	-3.873817	-1.154000
-0.502218876	1.136656	1.162366	-3.660148	127.908486	42.636162	0.920713	-0.545468	1.262464	-3.975342	-1.284164
-0.657710346	1.986365	1.260342	-3.602632	123.750255	41.250085	0.876979	-0.749973	1.437140	-4.108002	-1.496591
-0.700207538	2.228189	1.285795	-3.585588	122.710250	40.903417	0.866332	-0.808244	1.484184	-4.138817	-1.557047
-0.744472416	2.483519	1.311709	-3.567237	121.672466	40.557489	0.855817	-0.869896	1.532698	-4.168222	-1.620880
-0.839099631	3.038763	1.365073	-3.525973	119.608774	39.869591	0.835228	-1.004635	1.634371	-4.221568	-1.759691
-0.943451341	3.660889	1.420726	-3.477274	117.570530	39.190177	0.815295	-1.157191	1.742592	-4.265052	-1.915222
-1.00	4.000000	1.449490	-3.449490	116.565051	38.855017	0.805603	-1.241306	1.799261	-4.281874	-2.000000
-1.059938076	4.359413	1.478909	-3.418971	115.571231	38.523744	0.796113	-1.331392	1.857662	-4.294581	-2.089853
-1.12369091	4.740253	1.508992	-3.385301	114.591327	38.197109	0.786841	-1.428105	1.917786	-4.302397	-2.185063

FOR 3θ= 135°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	5.143471	1.539731	-3.347977	113.627965	37.875988	0.777806	-1.532199	1.979582	-4.304386	-2.285868
-1.20	5.192000	1.543358	-3.343358	113.516797	37.838932	0.776768	-1.544862	1.986896	-4.304189	-2.298000
-1.30	5.773000	1.585672	-3.285672	112.258839	37.419613	0.765100	-1.699124	2.072503	-4.294434	-2.443250
-1.40	6.336000	1.624871	-3.224871	111.156283	37.052094	0.754981	-1.854352	2.152203	-4.271463	-2.584000
-1.60	7.384000	1.693742	-3.093742	109.359963	36.453321	0.738701	-2.165964	2.292865	-4.188084	-2.846000
-1.70	7.857000	1.723289	-3.023289	108.642001	36.214000	0.732265	-2.321565	2.353369	-4.128683	-2.964250
-1.732050808	8.000000	1.732051	-3.000000	108.434949	36.144983	0.730416	-2.371322	2.371322	-4.107250	-3.000000
-1.741341639	8.040618	1.734526	-2.993184	108.376945	36.125648	0.729898	-2.385731	2.376393	-4.100822	-3.010154
-1.771714948	8.170700	1.742410	-2.970696	108.193532	36.064511	0.728264	-2.432792	2.392553	-4.079145	-3.042675
-1.828427125	8.402020	1.756281	-2.927854	107.875970	35.958657	0.725441	-2.520437	2.420986	-4.035967	-3.100505
-1.9196940611	8.740253	1.776228	-2.856534	107.430490	35.810163	0.721492	-2.660727	2.461881	-3.959202	-3.185063
-2.00	9.000000	1.791288	-2.791288	107.102729	35.700910	0.718597	-2.783201	2.492757	-3.884358	-3.250000
-2.10	9.269000	1.806657	-2.706657	106.775670	35.591890	0.715716	-2.934126	2.524266	-3.781749	-3.317250
-2.20	9.472000	1.818107	-2.618107	106.536792	35.512264	0.713616	-3.082890	2.547739	-3.668790	-3.368000
-2.267949192	9.569219	1.823547	-2.555598	106.424717	35.474906	0.712632	-3.182495	2.558889	-3.586138	-3.392305
-2.747477419	9.148632	1.799808	-2.052330	106.920503	35.640168	0.716991	-3.831957	2.510225	-2.862423	-3.287158
-2.886751345	8.603993	1.768239	-1.881488	107.607343	35.869114	0.723058	-3.992420	2.445501	-2.602126	-3.150998
-3.100131380471	7.338050	1.690823	-1.590692	109.432554	36.477518	0.739354	-4.193027	2.286893	-2.151462	-2.834513
-3.732050808	0.000000	1.000000	-0.267949	**135**	**45**	**1.000000**	-3.732051	1.000000	-0.267949	**-1.000000**
-3.8284271247	-1.656854	0.414214	0.414214	149.638807	49.879602	1.186680	-3.226165	0.349052	0.349052	-0.585786
-3.9953558031	-4.902666	IMAG.	IMAG.	192.716724	64.238908	2.072189	-1.928084	IMAG.	IMAG.	0.225666
-5	-36.000000	IMAG.	IMAG.	262.874984	87.624995	24.110665	-0.207377	IMAG.	IMAG.	8.000000
-5.67128182	-69.903760	IMAG.	IMAG.	266.526718	88.842239	49.481720	-0.114614	IMAG.	IMAG.	16.475940
-5.732050808	-73.569219	IMAG.	IMAG.	266.709305	88.903102	52.227979	-0.109751	IMAG.	IMAG.	17.392305
-5.846259147	-80.742791	IMAG.	IMAG.	267.016320	89.005440	57.603391	-0.101492	IMAG.	IMAG.	19.185698
-10	-671.000000	IMAG.	IMAG.	269.656401	89.885467	500.255331	-0.019990	IMAG.	IMAG.	166.750000
-15	-2656.000000	IMAG.	IMAG.	269.913581	89.971194	1989.001341	-0.007541	IMAG.	IMAG.	663.000000
-17.16933693	-4126.415996	IMAG.	IMAG.	269.944406	89.981469	3091.812860	-0.005553	IMAG.	IMAG.	1030.603999
-17.18882278704	-4141.601090	IMAG.	IMAG.	269.944610	89.981537	3103.201677	-0.005539	IMAG.	IMAG.	1034.400273

FOR 3θ= 120°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	6560.468407	IMAG.	IMAG.	90.034897	30.011632	0.577621	29.757960	IMAG.	IMAG.	-1641.849153
17.16933693	6539.796004	IMAG.	IMAG.	90.035007	30.011669	0.577622	29.724182	IMAG.	IMAG.	-1636.681052
14	3718.713824	IMAG.	IMAG.	90.061515	30.020505	0.577828	24.228682	IMAG.	IMAG.	-931.410507
11.45945741782	2151.090342	IMAG.	IMAG.	90.106201	30.035400	0.578174	19.820072	IMAG.	IMAG.	-539.504636
11.4300523	2136.126524	IMAG.	IMAG.	90.106942	30.035647	0.578180	19.769016	IMAG.	IMAG.	-535.763682
10	1487.883191	IMAG.	IMAG.	90.153319	30.051106	0.578540	17.284884	IMAG.	IMAG.	-373.702849
8.595086218	991.317439	IMAG.	IMAG.	90.229585	30.076528	0.579133	14.841311	IMAG.	IMAG.	-249.561410
8.555546781	979.189671	IMAG.	IMAG.	90.232408	30.077469	0.579154	14.772478	IMAG.	IMAG.	-246.529469
7	574.879418	IMAG.	IMAG.	90.393909	30.131303	0.580410	12.060442	IMAG.	IMAG.	-145.451905
5.8462591466	358.144952	IMAG.	IMAG.	90.627748	30.209249	0.582230	10.041150	IMAG.	IMAG.	-91.268289
5.732050809	340.133284	IMAG.	IMAG.	90.660324	30.220108	0.582484	9.840704	IMAG.	IMAG.	-86.765372
5.70	335.183941	IMAG.	IMAG.	90.669876	30.223292	0.582558	9.784429	IMAG.	IMAG.	-85.528036
5.671281820	330.788146	IMAG.	IMAG.	90.678594	30.226198	0.582626	9.733997	IMAG.	IMAG.	-84.429087
5.50	305.326560	IMAG.	IMAG.	90.733922	30.244641	0.583057	9.433033	IMAG.	IMAG.	-78.063691
5.00	238.171760	IMAG.	IMAG.	90.934977	30.311659	0.584626	8.552479	IMAG.	IMAG.	-61.274991
4.3032012473	161.263016	IMAG.	IMAG.	91.362377	30.454126	0.587967	7.318779	IMAG.	IMAG.	-42.047805
4.219331772	153.231565	IMAG.	IMAG.	91.430668	30.476889	0.588502	7.169615	IMAG.	IMAG.	-40.039942
3.8284271247	119.054615	IMAG.	IMAG.	91.818551	30.606184	0.591544	6.471922	IMAG.	IMAG.	-31.495705
3.732050808	111.425626	IMAG.	IMAG.	91.935687	30.645229	0.592464	6.299199	IMAG.	IMAG.	-29.588457
3.44991420939	90.822897	IMAG.	IMAG.	92.343251	30.781084	0.595672	5.791632	IMAG.	IMAG.	-24.437775
3.340232616	83.489013	IMAG.	IMAG.	92.533077	30.844359	0.597169	5.593442	IMAG.	IMAG.	-22.604304
3.141756715	71.143139	IMAG.	IMAG.	92.932995	30.977665	0.600330	5.233381	IMAG.	IMAG.	-19.517836
3.017830135	64.021652	IMAG.	IMAG.	93.226797	31.075599	0.602658	5.007535	IMAG.	IMAG.	-17.737464
2.886751345	56.965227	IMAG.	IMAG.	93.582284	31.194095	0.605481	4.767702	IMAG.	IMAG.	-15.973357
2.747477419	49.989057	IMAG.	IMAG.	94.019992	31.339997	0.608966	4.511708	IMAG.	IMAG.	-14.229315
2.267949192	29.856406	-3.732051	-3.732051	96.206023	32.068674	0.626537	3.619816	-5.956631	-5.956631	-9.196152
2.00	21.052559	-2.351542	-4.844610	98.135617	32.711872	0.642281	3.113901	-3.661234	-7.542818	-6.995191
1.8284271247	16.266861	-1.933604	-5.090975	99.784448	33.261483	0.655915	2.787596	-2.947948	-7.761635	-5.798766
1.732050808	13.856406	-1.732051	-5.196152	100.893395	33.631132	0.665182	2.603875	-2.603875	-7.811626	-5.196152
1.50	8.834292	-1.307149	-5.389004	104.239196	34.746399	0.693632	2.162531	-1.884500	-7.769260	-3.940624
1.40	6.996408	-1.143385	-5.452767	106.027303	35.342434	0.709152	1.974189	-1.612328	-7.689138	-3.481153

FOR 3θ= 120°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	5.346447	-0.988651	-5.507501	108.049488	36.016496	0.726983	1.788214	-1.359938	-7.575837	-3.068663
1.191753595	3.765282	-0.829947	-5.557959	110.508789	36.836263	0.749083	1.590950	-1.107950	-7.419683	-2.673371
1.10	2.586294	-0.701729	-5.594424	112.802425	37.600808	0.770126	1.428337	-0.911187	-7.264295	-2.378624
1.00	1.464102	-0.567874	-5.628279	115.483735	38.494578	0.795281	1.257417	-0.714054	-7.077090	-2.098076
0.90	0.505833	-0.439584	-5.656568	118.283172	39.427724	0.822220	1.094598	-0.534631	-6.879629	-1.858509
0.839099631	0.000000	-0.363970	-5.671282	**120**	**40**	**0.839100**	1.000000	-0.433763	-6.758770	**-1.732051**
0.657710346	-1.172901	-0.149022	-5.704841	124.799736	41.599912	0.887839	0.740799	-0.167848	-6.425537	-1.438826
0.502219976	-1.801439	0.024144	-5.722516	127.962041	42.654014	0.921288	0.545128	0.026207	-6.211427	-1.281691
0.466307658	-1.899712	0.062801	-5.725261	128.501101	42.833700	0.927103	0.502973	0.067739	-6.175429	-1.257123
0.431357893	-1.979016	0.099962	-5.727472	128.945610	42.981870	0.931924	0.462868	0.107264	-6.145860	-1.237297
0.363970234	-2.087388	0.170366	-5.730489	129.567164	43.189055	0.938703	0.387737	0.181491	-6.104687	-1.210204
0.299380347	-2.137635	0.236353	-5.731885	129.861004	43.287001	0.941924	0.317839	0.250925	-6.085294	-1.197642
0.267949192	-2.143594	0.267949	-5.732051	129.896091	43.298697	0.942309	0.284354	0.284354	-6.082982	-1.196152
0.237004353	-2.137878	0.298735	-5.731892	129.862432	43.287477	0.941940	0.251613	0.317149	-6.085200	-1.197581
0.20648339	-2.121157	0.328792	-5.731427	129.764245	43.254748	0.940862	0.219462	0.349458	-6.091675	-1.201761
0.17632698	-2.093995	0.358193	-5.730672	129.605594	43.201865	0.939124	0.187757	0.381412	-6.102148	-1.208552
0.00	-1.732051	**0.524423**	**-5.720575**	127.589089	42.529696	0.917285	0.000000	0.571712	-6.236420	-1.299038
-0.10	-1.381089	0.614566	-5.710719	125.795211	41.931737	0.898249	-0.111328	0.684183	-6.357612	-1.386778
-0.17632698	-1.046997	0.681449	-5.701275	124.220898	41.406966	0.881835	-0.199955	0.772763	-6.465242	-1.470301
-0.267949192	-0.574374	0.759608	-5.687811	122.192124	40.730708	0.861069	-0.311182	0.882169	-6.605526	-1.588457
-0.363970234	0.000000	0.839100	-5.671282	**120**	**40**	**0.839100**	-0.433763	1.000000	-6.758770	**-1.732051**
-0.40	0.235334	0.868303	-5.664455	119.178266	39.726089	0.830985	-0.481356	1.044907	-6.816551	-1.790884
-0.502218876	0.958528	0.949340	-5.643273	116.893258	38.964419	0.808756	-0.620977	1.173826	-6.977717	-1.971683
-0.657710346	2.204333	1.067595	-5.606038	113.653152	37.884384	0.778041	-0.845341	1.372158	-7.205323	-2.283134
-0.700207538	2.572891	1.098888	-5.594833	112.831293	37.610431	0.770394	-0.908896	1.426398	-7.262303	-2.375274
-0.744472416	2.968662	1.131021	-5.582701	112.007002	37.335667	0.762780	-0.975999	1.482762	-7.318887	-2.474216
-0.839099631	3.852998	1.198155	-5.555208	110.355669	36.785223	0.747693	-1.122251	1.602469	-7.429794	-2.695300
-0.943451341	4.883634	1.269762	-5.522463	108.708328	36.236109	0.732858	-1.287360	1.732618	-7.535520	-2.952959
-1.00	5.464102	1.307519	-5.503671	107.889091	35.963030	0.725557	-1.378251	1.802089	-7.585441	-3.098076
-1.059938076	6.094671	1.346743	-5.482957	107.074437	35.691479	0.718347	-1.475523	1.874779	-7.632737	-3.255719
-1.12369091	6.781243	1.387572	-5.460033	106.265641	35.421880	0.711238	-1.579909	1.950925	-7.676803	-3.427361

FOR 3θ= 120°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	7.530564	1.430155	-5.434554	105.464039	35.154680	0.704238	-1.692259	2.030783	-7.716923	-3.614692
-1.20	7.622409	1.435245	-5.431397	105.371059	35.123686	0.703430	-1.705928	2.040353	-7.721309	-3.637653
-1.30	8.752447	1.495759	-5.391911	104.310472	34.770157	0.694246	-1.872535	2.154508	-7.766572	-3.920163
-1.40	9.908408	1.554069	-5.350221	103.364422	34.454807	0.686120	-2.040459	2.265010	-7.797790	-4.209153
-1.60	12.274099	1.664137	-5.260290	101.766917	33.922306	0.672537	-2.379050	2.474416	-7.821558	-4.800576
-1.70	13.471830	1.715914	-5.212067	101.093706	33.697902	0.666864	-2.549245	2.573109	-7.815784	-5.100008
-1.732050808	13.856406	1.732051	-5.196152	100.893395	33.631132	0.665182	-2.603875	2.603875	-7.811626	-5.196152
-1.741341639	13.967896	1.736687	-5.191498	100.836654	33.612218	0.664706	-2.619718	2.612715	-7.810218	-5.224025
-1.771714948	14.332314	1.751713	-5.176150	100.655196	33.551732	0.663185	-2.671525	2.641364	-7.804989	-5.315129
-1.828427125	15.012027	1.779233	-5.146959	100.332315	33.444105	0.660484	-2.768316	2.693834	-7.792713	-5.485058
-1.9196940611	16.101519	1.822059	-5.098517	99.853324	33.284441	0.656489	-2.924185	2.775461	-7.766346	-5.757430
-2.00	17.052559	1.858244	-5.054397	99.469776	33.156592	0.653300	-3.061380	2.844396	-7.736715	-5.995191
-2.10	18.221981	1.901338	-4.997491	99.036893	33.012298	0.649713	-3.232197	2.926429	-7.691846	-6.287546
-2.20	19.369327	1.942239	-4.938391	98.648713	32.882904	0.646506	-3.402908	3.004209	-7.638587	-6.574383
-2.267949192	20.133284	1.968770	-4.896974	98.408096	32.802699	0.644523	-3.518804	3.054618	-7.597830	-6.765372
-2.747477419	24.994528	2.126287	-4.574962	97.142084	32.380695	0.634147	-4.332558	3.352988	-7.214358	-7.980683
-2.886751345	26.173212	2.161924	-4.471325	96.890257	32.296752	0.632094	-4.566962	3.420255	-7.073825	-8.275354
-3.100131380471	27.712813	2.207180	-4.303201	96.586776	32.195592	0.629626	-4.923765	3.505541	-6.834534	-8.660254
-3.732050808	29.856406	2.267949	-3.732051	96.206023	32.068674	0.626537	-5.956631	3.619816	-5.956631	-9.196152
-3.8284271247	29.799781	2.266375	-3.634101	96.215516	32.071839	0.626614	-6.109705	3.616860	-5.799583	-9.181996
-3.9953558031	29.422175	2.255838	-3.456635	96.279570	32.093190	0.627133	-6.370825	3.597064	-5.511804	-9.087594
-5	18.171760	1.899518	-2.095670	99.054675	33.018225	0.649860	-7.693966	2.922965	-3.224803	-6.274991
-5.67128182	0.000000	0.839100	-0.363970	**120**	**40**	**0.839100**	-6.758770	1.000000	-0.433763	**-1.732051**
-5.732050808	-2.143594	0.267949	0.267949	129.896091	43.298697	0.942309	-6.082982	0.284354	0.284354	-1.196152
-5.846259147	-6.413106	IMAG.	IMAG.	211.101593	70.367198	2.803247	-2.085531	IMAG.	IMAG.	0.603276
-10	-452.116809	IMAG.	IMAG.	269.488577	89.829526	336.095541	-0.029753	IMAG.	IMAG.	112.029202
-15	-2162.597756	IMAG.	IMAG.	269.893828	89.964609	1618.949964	-0.009265	IMAG.	IMAG.	539.649439
-17.16933693	-3479.752772	IMAG.	IMAG.	269.934062	89.978021	2606.815602	-0.006586	IMAG.	IMAG.	868.938193
-17.18882278704	-3493.467546	IMAG.	IMAG.	269.934321	89.978107	2617.101678	-0.006568	IMAG.	IMAG.	872.366886

FOR 3θ=100°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	10048.133104	IMAG.	IMAG.	90.022757	30.007586	0.577527	29.762814	IMAG.	IMAG.	-2517.704558
17.16933693	10019.548776	IMAG.	IMAG.	90.022822	30.007607	0.577527	29.729048	IMAG.	IMAG.	-2510.558476
14	6031.042428	IMAG.	IMAG.	90.037858	30.012619	0.577644	24.236382	IMAG.	IMAG.	-1513.431889
11.45945741782	3699.040685	IMAG.	IMAG.	90.061580	30.020527	0.577828	19.831951	IMAG.	IMAG.	-930.431453
11.4300523	3676.122747	IMAG.	IMAG.	90.061961	30.020654	0.577831	19.780961	IMAG.	IMAG.	-924.701969
10	2665.713264	IMAG.	IMAG.	90.085249	30.028416	0.578012	17.300687	IMAG.	IMAG.	-672.099598
8.595086218	1860.416274	IMAG.	IMAG.	90.121705	30.040568	0.578295	14.862813	IMAG.	IMAG.	-470.775350
8.555546781	1840.274617	IMAG.	IMAG.	90.123021	30.041007	0.578305	14.794179	IMAG.	IMAG.	-465.739936
7	1150.007146	IMAG.	IMAG.	90.195433	30.065144	0.578867	12.092583	IMAG.	IMAG.	-293.173068
5.8462591466	758.119650	-11.430052	-11.430052	90.293519	30.097840	0.579629	10.086203	-19.719589	-19.719589	-195.201194
5.732050809	724.480979	-9.971764	-12.774132	90.306734	30.102245	0.579732	9.887414	-17.200643	-22.034545	-186.791527
5.70	715.201557	-9.772380	-12.941465	90.310591	30.103530	0.579762	9.831620	-16.855847	-22.322029	-184.471671
5.671281820	706.946546	-9.610513	-13.074614	90.314105	30.104702	0.579789	9.781624	-16.57587	-22.550628	-182.407918
5.50	658.872543	-8.829546	-13.684300	90.336260	30.112087	0.579962	9.483387	-15.224363	-23.595183	-170.389418
5.00	529.674855	-7.254174	-14.759671	90.414909	30.138303	0.580573	8.612178	-12.494848	-25.422583	-138.089995
4.3032012473	376.158400	-5.671282	-15.645765	90.574600	30.191533	0.581816	7.396154	-9.747551	-26.891255	-99.710882
4.219331772	359.679894	-5.505607	-15.727570	90.599361	30.199787	0.582009	7.249600	-9.459661	-27.022902	-95.591255
3.8284271247	288.325588	-4.781370	-16.060902	90.736857	30.245619	0.583080	6.565866	-8.200192	-27.544923	-77.752679
3.732050808	272.085625	-4.613320	-16.132576	90.777448	30.259149	0.583397	6.397106	-7.907689	-27.652837	-73.692688
3.44991420939	227.536762	-4.141465	-16.322294	90.915842	30.305281	0.584476	5.902572	-7.085769	-27.926350	-62.555472
3.340232616	211.401603	-3.965375	-16.388703	90.978957	30.326319	0.584969	5.710100	-6.778776	-28.016352	-58.521682
3.141756715	183.852046	-3.656113	-16.499489	91.109507	30.369836	0.585989	5.361460	-6.239217	-28.156651	-51.634293
3.017830135	167.709681	-3.468715	-16.562960	91.203549	30.401183	0.586724	5.143524	-5.912003	-28.229547	-47.598702
2.886751345	151.506771	-3.274943	-16.625653	91.315462	30.438487	0.587600	4.912785	-5.573425	-28.294177	-43.547975
2.747477419	135.257243	-3.073777	-16.687546	91.450745	30.483582	0.588659	4.667349	-5.221658	-28.348404	-39.485593
2.267949192	86.702585	-2.414780	-16.867015	92.094212	30.698071	0.593711	3.819955	-4.067265	-28.409469	-27.346928
2.00	64.384100	-2.066903	-16.946942	92.630345	30.876782	0.597937	3.344832	-3.456722	-28.342335	-21.767307
1.8284271247	51.835900	-1.851058	-16.991215	93.072467	31.024156	0.601435	3.040110	-3.077738	-28.251144	-18.630257
1.732050808	45.370255	-1.732051	-17.013845	93.363727	31.121242	0.603744	2.868848	-2.868848	-28.180546	-17.013845
1.50	31.484870	-1.451806	-17.062040	94.223148	31.407716	0.610587	2.456650	-2.377720	-27.943645	-13.542499
1.40	26.219855	-1.333674	-17.080171	94.675885	31.558628	0.614209	2.279354	-2.171368	-27.808394	-12.226246

FOR 3θ=100°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	21.379117	-1.217072	-17.096774	95.186897	31.728966	0.618311	2.102501	-1.968380	-27.650759	-11.016061
1.191753595	16.610444	-1.092533	-17.113066	95.812269	31.937423	0.623352	1.911848	-1.752675	-27.453305	-9.823893
1.10	12.946471	-0.988303	-17.125542	96.405202	32.135067	0.628152	1.751168	-1.573351	-27.263372	-8.907900
1.00	9.342564	-0.876067	-17.137778	97.118919	32.372973	0.633958	1.577392	-1.381901	-27.032995	-8.006923
0.90	6.138933	-0.765219	-17.148626	97.900646	32.633549	0.640352	1.405477	-1.194998	-26.779999	-7.206015
0.839099631	4.381467	-0.698380	-17.154565	98.406532	32.802177	0.644510	1.301919	-1.083583	-26.616454	-6.766649
0.657710346	0.000000	-0.502219	-17.169337	**100**	**33 1/3**	**0.657710**	1.000000	-0.763587	-26.104709	**-5.671282**
0.502219976	-2.759954	-0.337450	-17.178616	101.351305	33.783768	0.669031	0.750667	-0.504385	-25.676843	-4.981293
0.466307658	-3.269271	-0.299827	-17.180326	101.641045	33.880348	0.671474	0.694453	-0.446521	-25.585974	-4.853964
0.431357893	-3.719333	-0.263367	-17.181837	101.909488	33.969829	0.673743	0.640241	-0.390901	-25.502077	-4.741449
0.363970234	-4.461078	-0.193491	-17.184325	102.379554	34.126518	0.677726	0.537046	-0.285500	-25.355855	-4.556012
0.299380347	-5.017663	-0.127035	-17.186191	102.756975	34.252325	0.680935	0.439661	-0.186559	-25.239101	-4.416866
0.267949192	-5.234352	-0.094877	-17.186918	102.910104	34.303368	0.682240	0.392749	-0.139067	-25.191898	-4.362694
0.237004353	-5.413296	-0.063332	-17.187517	103.039310	34.346437	0.683342	0.346831	-0.092680	-25.152145	-4.317958
0.20648339	-5.556537	-0.032331	-17.187997	103.144582	34.381527	0.684241	0.301770	-0.047251	-25.119807	-4.282148
0.17632698	-5.665800	-0.001809	-17.188364	103.226012	34.408671	0.684937	0.257435	-0.002641	-25.094824	-4.254832
0.00	-5.671282	0.174536	-17.188382	103.230124	34.410041	0.684972	0.000000	0.254808	-25.093564	-4.253461
-0.10	-5.202143	0.272964	-17.186810	102.887115	34.295705	0.682044	-0.146618	0.400215	-25.198978	-4.370746
-0.17632698	-4.618802	0.347335	-17.184854	102.484257	34.161419	0.678615	-0.259833	0.511830	-25.323408	-4.516581
-0.267949192	-3.665133	0.435758	-17.181655	101.876515	33.958838	0.673464	-0.397867	0.647041	-25.512367	-4.754999
-0.363970234	-2.373690	0.527443	-17.177318	101.140887	33.713629	0.667261	-0.545469	0.790460	-25.743035	-5.077859
-0.40	-1.813067	0.561589	-17.175435	100.848838	33.616279	0.664808	-0.601677	0.844739	-25.835179	-5.218015
-0.502218876	0.000000	0.657710	-17.169337	**100**	**33 1/3**	**0.657710**	-0.763587	1.000000	-26.104709	**-5.671282**
-0.657710346	3.377234	0.801820	-17.157956	98.725557	32.908519	0.647140	-1.016334	1.239022	-26.513516	-6.515590
-0.700207538	4.427764	0.840771	-17.154409	98.392382	32.797461	0.644393	-1.086615	1.304748	-26.621023	-6.778223
-0.744472416	5.579259	0.881145	-17.150518	98.055056	32.685019	0.641619	-1.160302	1.373315	-26.730049	-7.066097
-0.839099631	8.234465	0.966788	-17.141534	97.371290	32.457097	0.636018	-1.319302	1.520063	-26.951332	-7.729898
-0.943451341	11.463337	1.060188	-17.130582	96.680929	32.226976	0.630391	-1.496612	1.681793	-27.174520	-8.537116
-1.00	13.342564	1.110350	-17.124195	96.335358	32.111786	0.627586	-1.593408	1.769241	-27.285836	-9.006923
-1.059938076	15.432248	1.163175	-17.117082	95.990637	31.996879	0.624794	-1.696461	1.861694	-27.396378	-9.529344
-1.12369091	17.763991	1.218976	-17.109131	95.647599	31.882533	0.622022	-1.806512	1.959698	-27.505655	-10.112280

FOR 3θ=100°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	20.375726	1.278115	-17.100207	95.307078	31.769026	0.619278	-1.924424	2.063879	-27.613131	-10.765213
-1.20	20.700656	1.285250	-17.099096	95.267556	31.755852	0.618960	-1.938736	2.076467	-27.625524	-10.846446
-1.30	24.785117	1.371253	-17.085099	94.816554	31.605518	0.615337	-2.112664	2.228460	-27.765440	-11.867561
-1.40	29.131855	1.456307	-17.070153	94.414180	31.471393	0.612114	-2.287155	2.379143	-27.887201	-12.954246
-1.60	38.588163	1.623608	-17.037454	93.735043	31.245014	0.606696	-2.637236	2.676149	-28.082364	-15.318322
-1.70	43.685732	1.705876	-17.019722	93.448897	31.149632	0.604421	-2.812611	2.822333	-28.158736	-16.592715
-1.732050808	45.370255	1.732051	-17.013845	93.363727	31.121242	0.603744	-2.868848	2.868848	-28.180546	-17.013845
-1.741341639	45.863109	1.739621	-17.012125	93.339598	31.113199	0.603553	-2.885152	2.882301	-28.186638	-17.137059
-1.771714948	47.488512	1.764314	-17.006445	93.262413	31.087471	0.602940	-2.938458	2.926183	-28.205849	-17.543410
-1.828427125	50.581066	1.810198	-16.995616	93.124986	31.041662	0.601851	-3.038008	3.007719	-28.238924	-18.316548
-1.9196940611	55.713149	1.883434	-16.977585	92.920786	30.973595	0.600234	-3.198245	3.137834	-28.284965	-19.599569
-2.00	60.384100	1.947260	-16.961106	92.756812	30.918937	0.598937	-3.339251	3.251196	-28.318698	-20.767307
-2.10	66.398777	2.025945	-16.939790	92.570939	30.856980	0.597468	-3.514831	3.390882	-28.352616	-22.270976
-2.20	72.627730	2.103754	-16.917599	92.403125	30.801042	0.596144	-3.690382	3.528934	-28.378363	-23.828214
-2.267949192	76.979463	2.156129	-16.902025	92.298311	30.766104	0.595318	-3.809643	3.621810	-28.391587	-24.916147
-2.747477419	110.262715	2.514553	-16.780921	91.723337	30.574446	0.590797	-4.650463	4.256209	-28.403894	-33.236961
-2.886751345	120.714756	2.615043	-16.742137	91.597796	30.532599	0.589812	-4.894361	4.433692	-28.385566	-35.849971
-3.100131380471	137.351238	2.765910	-16.679624	91.431771	30.477257	0.588510	-5.267759	4.699849	-28.342103	-40.009091
-3.732050808	190.516406	3.191185	-16.472980	91.074834	30.358278	0.585718	-6.371753	5.448330	-28.124417	-53.300383
-3.8284271247	199.070754	3.253266	-16.438684	91.033381	30.344460	0.585394	-6.539913	5.557393	-28.081391	-55.438970
-3.9953558031	214.127217	3.359077	-16.377566	90.967692	30.322564	0.584881	-6.831055	5.743177	-28.001525	-59.203086
-5	309.674855	3.950604	-15.964450	90.689530	30.229843	0.582711	-8.580577	6.779693	-27.396838	-83.089995
-5.67128182	376.158400	4.303201	-15.645765	90.574600	30.191533	0.581816	-9.747551	7.396154	-26.891255	-99.710882
-5.732050808	382.204101	4.333447	-15.615242	90.566021	30.188674	0.581749	-9.853129	7.448994	-26.841875	-101.222307
-5.846259147	393.561592	4.389539	-15.557125	90.550577	30.183526	0.581629	-10.051526	7.546973	-26.747504	-104.061680
-10	725.713264	5.736289	-12.750134	90.306229	30.102076	0.579728	-17.249465	9.894791	-21.993299	-187.099598
-15	492.443947	4.842998	-6.856844	90.444895	30.148298	0.580807	-25.826157	8.338402	-11.805728	-128.782268
-17.16933693	0.000000	0.657710	-0.502219	**100**	**33 1/3**	**0.657710**	-26.104709	1.000000	-0.763587	-5.671282
-17.18882278704	-5.802850	0.087489	0.087489	103.329563	34.443188	0.685822	-25.063095	0.127568	0.127568	-4.220569

FOR 3θ=80°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	5.802850	-0.087489	-0.087489	76.670437	25.556812	0.478193	35.945347	-0.182957	-0.182957	4.220569
17.16933693	0.000000	0.502219	-0.657710	**80**	**26 2/3**	**0.502219**	34.186961	1.000000	-1.309609	**5.671282**
14	-627.042428	8.396382	-5.382537	89.647267	29.882422	0.574617	24.364040	14.612128	-9.367167	162.431889
11.45945741782	-758.104686	11.400597	-5.846209	89.706475	29.902158	0.575076	19.926870	19.824518	-10.165983	195.197453
11.4300523	-758.119650	11.430052	-5.846259	89.706481	29.902160	0.575076	19.875736	19.875736	-10.166069	195.201194
10	-725.713264	12.750134	-5.736289	89.693771	29.897924	0.574977	17.391991	22.175023	-9.976549	187.099598
8.595086218	-642.054085	13.857117	-5.438358	89.655233	29.885078	0.574679	14.956326	24.112796	-9.463297	166.184803
8.555546781	-639.120667	13.885809	-5.427510	89.653705	29.884568	0.574667	14.887829	24.163220	-9.444615	165.451449
7	-506.007146	14.914808	-4.900963	89.566518	29.855506	0.573993	12.195279	25.984320	-8.538372	132.173068
5.8462591466	-393.561592	15.557125	-4.389539	89.449423	29.816474	0.573087	10.201341	27.146170	-7.659459	104.061680
5.732050809	-382.204101	15.615242	-4.333447	89.433979	29.811326	0.572968	10.004139	27.253255	-7.563158	101.222307
5.70	-379.015557	15.631375	-4.317529	89.429487	29.809829	0.572933	9.948803	27.283065	-7.535833	100.425171
5.671281820	-376.158400	15.645765	-4.303201	89.425400	29.808467	0.572902	9.899224	27.309687	-7.511239	99.710882
5.50	-359.122543	15.730303	-4.216457	89.399764	29.799921	0.572704	9.603572	27.466744	-7.362373	95.451918
5.00	-309.674855	15.964450	-3.950604	89.310470	29.770157	0.572014	8.741046	27.909199	-6.906483	83.089995
4.3032012473	-242.608197	16.259189	-3.548545	89.136180	29.712060	0.570669	7.540627	28.491457	-6.218220	66.323331
4.219331772	-234.764378	16.292164	-3.497651	89.109866	29.703289	0.570466	7.396290	28.559396	-6.131217	64.362376
3.8284271247	-199.070754	16.438684	-3.253266	88.966619	29.655540	0.569362	6.724066	28.872116	-5.713880	55.438970
3.732050808	-190.516406	16.472980	-3.191185	88.925166	29.641722	0.569043	6.558473	28.948586	-5.607990	53.300383
3.44991420939	-166.115123	16.569174	-3.005243	88.786290	29.595430	0.567974	6.074075	29.172440	-5.291167	47.200063
3.340232616	-156.908019	16.604865	-2.931252	88.724087	29.574696	0.567495	5.885924	29.259930	-5.165247	44.898287
3.141756715	-140.680317	16.666997	-2.794908	88.597394	29.532465	0.566521	5.545702	29.419910	-4.933459	40.841361
3.017830135	-130.848101	16.704176	-2.708161	88.507611	29.502537	0.565831	5.333446	29.521481	-4.786164	38.383307
2.886751345	-120.714756	16.742137	-2.615043	88.402204	29.467401	0.565022	5.109096	29.630950	-4.628215	35.849971
2.747477419	-110.262715	16.780921	-2.514553	88.276663	29.425554	0.564059	4.870906	29.750305	-4.457963	33.236961
2.267949192	-76.979463	16.902025	-2.156129	87.701689	29.233896	0.559658	4.052386	30.200645	-3.852585	24.916147
2.00	-60.384100	16.961106	-1.947260	87.243188	29.081063	0.556160	3.596087	30.496809	-3.501259	20.767307
1.8284271247	-50.581066	16.995616	-1.810198	86.875014	28.958338	0.553359	3.304234	30.713551	-3.271291	18.316548
1.732050808	-45.370255	17.013845	-1.732051	86.636273	28.878758	0.551546	3.140356	30.847553	-3.140356	17.013845
1.50	-33.734870	17.054267	-1.540422	85.944695	28.648232	0.546310	2.745693	31.217184	-2.819683	14.104999
1.40	-29.131855	17.070153	-1.456307	85.585820	28.528607	0.543602	2.575412	31.401911	-2.678993	12.954246

FOR 3θ=80°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-24.785117	17.085099	-1.371253	85.183446	28.394482	0.540574	2.404853	31.605504	-2.536664	11.867561
1.191753595	-20.375726	17.100207	-1.278115	84.692922	28.230974	0.536892	2.219729	31.850395	-2.380584	10.765213
1.10	-16.884471	17.112132	-1.198286	84.227710	28.075903	0.533410	2.062204	32.080640	-2.246464	9.892400
1.00	-13.342564	17.124195	-1.110350	83.664642	27.888214	0.529209	1.889611	32.358071	-2.098129	9.006923
0.90	-10.080933	17.135274	-1.021429	83.039911	27.679970	0.524566	1.715704	32.665629	-1.947189	8.191515
0.839099631	-8.234465	17.141534	-0.966788	82.628710	27.542903	0.521519	1.608953	32.868464	-1.853791	7.729898
0.657710346	-3.377234	17.157956	-0.801820	81.274443	27.091481	0.511538	1.285750	33.541882	-1.567469	6.515590
0.502219976	-0.000021	17.169337	-0.657711	**80**	**26 2/3**	**0.502219**	1.000002	34.186956	-1.309611	**5.671287**
0.466307658	0.674215	17.171605	-0.624068	79.700153	26.566718	0.500036	0.932548	34.340714	-1.248045	5.502728
0.431357893	1.291711	17.173682	-0.591195	79.409494	26.469831	0.497924	0.866312	34.490545	-1.187318	5.348354
0.363970234	2.373690	17.177318	-0.527443	78.859113	26.286371	0.493935	0.736879	34.776486	-1.067840	5.077859
0.299380347	3.275047	17.180345	-0.465880	78.355586	26.118529	0.490296	0.610611	35.040761	-0.950202	4.852520
0.267949192	3.665133	17.181655	-0.435758	78.123485	26.041162	0.488622	0.548377	35.163475	-0.891810	4.754999
0.237004353	4.017896	17.182838	-0.405997	77.905598	25.968533	0.487053	0.486609	35.279202	-0.833579	4.666808
0.20648339	4.335243	17.183903	-0.376541	77.702748	25.900916	0.485594	0.425218	35.387412	-0.775424	4.587471
0.17632698	4.618802	17.184854	-0.347335	77.515743	25.838581	0.484250	0.364124	35.487574	-0.717265	4.516581
0.00	5.671282	**17.188382**	**-0.174536**	76.769876	25.589959	0.478904	0.000000	35.891061	-0.364450	4.253461
-0.10	5.800143	17.188814	-0.074968	76.672497	25.557499	0.478208	-0.209114	35.944222	-0.156769	4.221246
-0.17632698	5.665800	17.188364	0.001809	76.773988	25.591329	0.478934	-0.368166	35.888819	0.003777	4.254832
-0.267949192	5.234352	17.186918	0.094877	77.089896	25.696632	0.481195	-0.556841	35.717150	0.197169	4.362694
-0.363970234	4.461078	17.184325	0.193491	77.620446	25.873482	0.485002	-0.750451	35.431445	0.398948	4.556012
-0.40	4.085067	17.183064	0.230782	77.863214	25.954405	0.486748	-0.821781	35.301774	0.474130	4.650015
-0.502218876	2.759970	17.178616	0.337448	78.648686	26.216229	0.492413	-1.019914	34.886610	0.685296	4.981289
-0.657710346	0.000000	17.169337	0.502219	**80**	**26 2/3**	**0.502219**	-1.309609	34.186961	1.000000	**5.671282**
-0.700207538	-0.913129	17.166263	0.547790	80.379575	26.793192	0.504987	-1.386585	33.993460	1.084761	5.899564
-0.744472416	-1.937656	17.162811	0.595507	80.772840	26.924280	0.507862	-1.465895	33.794247	1.172577	6.155696
-0.839099631	-4.381467	17.154565	0.698380	81.593468	27.197823	0.513882	-1.632864	33.382295	1.359027	6.766649
-0.943451341	-7.482162	17.144081	0.813216	82.446984	27.482328	0.520175	-1.813719	32.958289	1.563350	7.541822
-1.00	-9.342564	17.137778	0.876067	82.881081	27.627027	0.523388	-1.910628	32.743918	1.673838	8.006923
-1.059938076	-11.454234	17.130613	0.943171	83.317305	27.772435	0.526626	-2.012698	32.529022	1.790970	8.534840
-1.12369091	-13.859573	17.122436	1.015100	83.753556	27.917852	0.529872	-2.120685	32.314304	1.915747	9.136175

FOR 3θ=80°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-16.610444	17.113066	1.092533	84.187731	28.062577	0.533111	-2.235469	32.100369	2.049353	9.823893
-1.20	-16.956656	17.111885	1.101960	84.238150	28.079383	0.533488	-2.249348	32.075487	2.065576	9.910446
-1.30	-21.379117	17.096774	1.217072	84.813103	28.271034	0.537793	-2.417289	31.790646	2.263087	11.016061
-1.40	-26.219855	17.080171	1.333674	85.324115	28.441372	0.541632	-2.584783	31.534670	2.462328	12.226246
-1.60	-37.180163	17.042339	1.571507	86.177368	28.725789	0.548069	-2.919340	31.095233	2.867351	14.966322
-1.70	-43.311732	17.021025	1.692820	86.531606	28.843869	0.550752	-3.086688	30.905056	3.073652	16.499215
-1.732050808	-45.370255	17.013845	1.732051	86.636273	28.878758	0.551546	-3.140356	30.847553	3.140356	17.013845
-1.741341639	-45.975498	17.011732	1.743455	86.665857	28.888619	0.551771	-3.155916	30.831173	3.159746	17.165156
-1.771714948	-47.980956	17.004722	1.780838	86.760273	28.920091	0.552487	-3.206798	30.778488	3.223311	17.666521
-1.828427125	-51.835900	16.991215	1.851058	86.927533	28.975844	0.553758	-3.301852	30.683463	3.342720	18.630257
-1.9196940611	-58.343995	16.968311	1.965228	87.173890	29.057963	0.555632	-3.454972	30.538743	3.536922	20.257280
-2.00	-64.384100	16.946942	2.066903	87.369655	29.123218	0.557124	-3.589867	30.418631	3.709954	21.767307
-2.10	-72.320777	16.918695	2.195150	87.589120	29.196373	0.558798	-3.758066	30.276941	3.928342	23.751476
-2.20	-80.723730	16.888578	2.325267	87.784823	29.261608	0.560293	-3.926517	30.142400	4.150091	25.852214
-2.267949192	-86.702585	16.867015	2.414780	87.905788	29.301929	0.561218	-4.041119	30.054295	4.302748	27.346928
-2.747477419	-135.257243	16.687546	3.073777	88.549255	29.516418	0.566151	-4.852905	29.475428	5.429251	39.485593
-2.886751345	-151.506771	16.625653	3.274943	88.684538	29.561513	0.567191	-5.089559	29.312271	5.773970	43.547975
-3.100131380471	-178.340026	16.521278	3.592698	88.860079	29.620026	0.568541	-5.452780	29.059056	6.319149	50.256288
-3.732050808	-272.085625	16.132576	4.613320	89.222552	29.740851	0.571335	-6.532156	28.236619	8.074629	73.692688
-3.8284271247	-288.325588	16.060902	4.781370	89.263143	29.754381	0.571649	-6.697170	28.095764	8.364178	77.752679
-3.9953558031	-317.709757	15.927460	5.081742	89.326745	29.775582	0.572140	-6.983184	27.838416	8.881996	85.098721
-5	-529.674855	14.759671	7.254174	89.585091	29.861697	0.574136	-8.708734	25.707609	12.634934	138.089995
-5.67128182	-706.946546	13.074614	9.610513	89.685895	29.895298	0.574916	-9.864534	22.741770	16.716368	182.407918
-5.732050808	-724.480979	12.774132	9.971764	89.693266	29.897755	0.574973	-9.969246	22.216910	17.343002	186.791527
-5.846259147	-758.119650	11.430052	11.430052	89.706481	29.902160	0.575076	-10.166069	19.875736	19.875736	195.201194
-10	-2665.713264	IMAG.	IMAG.	89.914751	29.971584	0.576689	-17.340363	IMAG.	IMAG.	672.099598
-15	-7152.443947	IMAG.	IMAG.	89.968059	29.989353	0.577103	-25.991915	IMAG.	IMAG.	1793.782268
-17.16933693	-10019.548776	IMAG.	IMAG.	89.977178	29.992393	0.577173	-29.747284	IMAG.	IMAG.	2510.558476
-17.18882278704	-10048.133104	IMAG.	IMAG.	89.977243	29.992414	0.577174	-29.781019	IMAG.	IMAG.	2517.704558

FOR 3θ= 70°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	2594.442449	IMAG.	IMAG.	-89.911288	-29.970429	-0.576662	-29.807431	IMAG.	IMAG.	-645.863135
17.16933693	2582.767154	IMAG.	IMAG.	-89.910885	-29.970295	-0.576659	-29.773802	IMAG.	IMAG.	-642.944311
14	1089.230755	IMAG.	IMAG.	-89.787448	-29.929149	-0.575703	-24.318108	IMAG.	IMAG.	-269.560211
11.45945741782	390.826161	IMAG.	IMAG.	-89.396649	-29.798883	-0.572680	-20.010245	IMAG.	IMAG.	-94.959063
11.4300523	384.907433	IMAG.	IMAG.	-89.387099	-29.795700	-0.572606	-19.961470	IMAG.	IMAG.	-93.479381
10	148.504252	IMAG.	IMAG.	-88.333857	-29.444619	-0.564497	-17.714870	IMAG.	IMAG.	-34.378586
8.595086218	3.014709	-0.176327	-0.176327	63.363727	21.121242	0.386294	22.250121	-0.456458	-0.456458	1.993800
8.555546781	0.000000	0.431358	-0.744472	**70**	**23 1/3**	**0.431358**	19.833987	1.000000	-1.725881	**2.747477**
7	-79.131703	4.097268	-2.854836	87.458625	29.152875	0.557802	12.549251	7.345377	-5.118007	22.530403
5.8462591466	-96.689492	5.492741	-3.096568	87.872594	29.290865	0.560964	10.421804	9.791608	-5.520082	26.919850
5.732050809	-96.930788	5.610089	-3.099707	87.877346	29.292449	0.561001	10.217549	10.000149	-5.525320	26.980174
5.70	-96.956147	5.642469	-3.100037	87.877845	29.292615	0.561004	10.160349	10.057799	-5.525870	26.986514
5.671281820	-96.963404	5.671282	-3.100131	87.877987	29.292662	0.561005	10.109139	10.109139	-5.526027	26.988329
5.50	-96.711098	5.839282	-3.096850	87.873021	29.291007	0.560967	9.804490	10.409306	-5.520551	26.925252
5.00	-93.313329	6.294599	-3.052166	87.803799	29.267933	0.560438	8.921591	11.231567	-5.446036	26.075810
4.3032012473	-83.106998	6.851284	-2.912053	87.565858	29.188619	0.558620	7.703265	12.264649	-5.212937	23.524227
4.219331772	-81.532813	6.912686	-2.889585	87.524495	29.174832	0.558305	7.557399	12.381563	-5.175641	23.130681
3.8284271247	-73.433234	7.183905	-2.769900	87.287333	29.095778	0.556496	6.879519	12.909168	-4.977391	21.105786
3.732050808	-71.270184	7.247070	-2.736688	87.216114	29.072038	0.555954	6.712879	13.035381	-4.922510	20.565024
3.44991420939	-64.642374	7.423767	-2.631249	86.972592	28.990864	0.554101	6.226151	13.397867	-4.748684	18.908071
3.340232616	-61.967816	7.489192	-2.586993	86.861828	28.953943	0.553259	6.037380	13.536512	-4.675919	18.239432
3.141756715	-57.024701	7.602991	-2.502315	86.634261	28.878087	0.551531	5.696431	13.785253	-4.537037	17.003653
3.017830135	-53.888025	7.671064	-2.446461	86.471938	28.823979	0.550300	5.483975	13.939790	-4.445689	16.219484
2.886751345	-50.543451	7.740582	-2.384901	86.280699	28.760233	0.548851	5.259625	14.103244	-4.345260	15.383340
2.747477419	-46.974348	7.811655	-2.316700	86.052389	28.684130	0.547124	5.021672	14.277667	-4.234324	14.491064
2.267949192	-34.786683	8.034363	-2.059880	85.006127	28.335376	0.539241	4.205816	14.899386	-3.819960	11.444148
2.00	-28.222252	8.143850	-1.901418	84.175453	28.058484	0.533019	3.752208	15.278711	-3.567258	9.803040
1.8284271247	-24.180758	8.208243	-1.794238	83.511566	27.837189	0.528070	3.462471	15.543855	-3.397728	8.792667
1.732050808	-21.979819	8.242432	-1.732051	83.082489	27.694163	0.524882	3.299887	15.703407	-3.299887	8.242432
1.50	-16.922995	8.318816	-1.576384	81.844870	27.281623	0.515732	2.908485	16.130103	-3.056593	6.978226
1.40	-14.863690	8.349109	-1.506676	81.205083	27.068361	0.511029	2.739569	16.337829	-2.948317	6.463400

FOR 3θ= 70°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-12.885233	8.377794	-1.435361	80.489095	26.829698	0.505787	2.570251	16.563873	-2.837877	5.968786
1.191753595	-10.841698	8.407008	-1.356329	79.617392	26.539131	0.499435	2.386205	16.833050	-2.715729	5.457902
1.10	-9.194866	8.430255	-1.287823	78.790968	26.263656	0.493442	2.229240	17.084600	-2.609878	5.046194
1.00	-7.494955	8.453983	-1.211551	77.789841	25.929947	0.486220	2.056682	17.387157	-2.491775	4.621216
0.90	-5.899893	8.476007	-1.133575	76.676164	25.558721	0.478234	1.881923	17.723549	-2.370334	4.222451
0.839099631	-4.982421	8.488572	-1.085239	75.940404	25.313468	0.472985	1.774050	17.946793	-2.294446	3.993083
0.657710346	-2.506675	8.522113	-0.937391	73.491689	24.497230	0.455668	1.443399	18.702466	-2.057180	3.374146
0.502219976	-0.711457	8.546110	-0.805898	71.127498	23.709166	0.439160	1.143592	19.460122	-1.835088	2.925342
0.466307658	-0.342308	8.551011	-0.774887	70.558195	23.519398	0.435215	1.071442	19.647786	-1.780469	2.833054
0.431357893	0.000000	8.555547	-0.744472	**70**	**23 1/3**	**0.431358**	1.000000	19.833987	-1.725881	**2.747477**
0.363970234	0.611873	8.563630	-0.685168	68.921873	22.973958	0.423939	0.858545	20.200171	-1.616197	2.594509
0.299380347	1.137412	8.570549	-0.627497	67.903417	22.634472	0.416966	0.717997	20.554555	-1.504912	2.463124
0.267949192	1.371088	8.573618	-0.599135	67.419951	22.473317	0.413668	0.647740	20.725841	-1.448348	2.404705
0.237004353	1.586791	8.576448	-0.571020	66.955546	22.318515	0.410507	0.577345	20.892307	-1.391010	2.350780
0.20648339	1.785411	8.579050	-0.543101	66.511674	22.170558	0.407493	0.506716	21.053238	-1.332786	2.301125
0.17632698	1.967712	8.581435	-0.515330	66.089851	22.029950	0.404634	0.435769	21.207873	-1.273570	2.255550
0.00	2.747477	**8.591610**	**-0.349178**	64.112996	21.370999	0.391312	0.000000	21.955912	-0.892326	2.060608
-0.10	2.964053	8.594428	-0.251995	63.508831	21.169610	0.387264	-0.258222	22.192663	-0.650707	2.006464
-0.17632698	3.014709	8.595086	-0.176327	63.363727	21.121242	0.386294	-0.456458	22.250121	-0.456458	1.993800
-0.267949192	2.940307	8.594119	-0.083738	63.576348	21.192116	0.387716	-0.691096	22.166008	-0.215976	2.012401
-0.363970234	2.699261	8.590982	0.015420	64.244026	21.414675	0.392191	-0.928043	21.905086	0.039318	2.072662
-0.40	2.564688	8.589229	0.053203	64.603270	21.534423	0.394605	-1.013673	21.766669	0.134826	2.106305
-0.502218876	2.048525	8.582492	0.162159	65.898263	21.966088	0.403338	-1.245157	21.278665	0.402043	2.235346
-0.657710346	0.870559	8.567038	0.333104	68.431991	22.810664	0.420580	-1.563816	20.369566	0.792011	2.529838
-0.700207538	0.463608	8.561674	0.380966	69.193184	23.064395	0.425802	-1.644445	20.107179	0.894702	2.631575
-0.744472416	0.000000	8.555547	0.431358	**70**	**23 1/3**	**0.431358**	-1.725881	19.833987	1.000000	**2.747477**
-0.839099631	-1.129423	8.540547	0.540985	71.734466	23.911489	0.443379	-1.892511	19.262410	1.220142	3.029833
-0.943451341	-2.598528	8.520878	0.665006	73.597263	24.532421	0.456410	-2.067115	18.669357	1.457037	3.397109
-1.00	-3.494955	8.508787	0.733645	74.562492	24.854164	0.463213	-2.158836	18.369077	1.583820	3.621216
-1.059938076	-4.523630	8.494828	0.807543	75.541806	25.180602	0.470151	-2.254464	18.068303	1.717625	3.878385
-1.12369091	-5.707878	8.478643	0.887480	76.528502	25.509501	0.477179	-2.354862	17.768262	1.859847	4.174447

FOR 3θ= 70°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-7.076415	8.459785	0.974401	77.515744	25.838581	0.484250	-2.461030	17.469874	2.012187	4.516581
-1.20	-7.249625	8.457386	0.985047	77.630633	25.876878	0.485075	-2.473843	17.435202	2.030709	4.559884
-1.30	-9.479233	8.426259	1.116173	78.942825	26.314275	0.494541	-2.628701	17.038552	2.256988	5.117286
-1.40	-11.951690	8.391191	1.251241	80.109574	26.703191	0.503017	-2.783204	16.681711	2.487471	5.735400
-1.60	-17.649149	8.308025	1.534407	82.048969	27.349656	0.517237	-3.093362	16.062333	2.966548	7.159765
-1.70	-20.886152	8.259203	1.683230	82.847567	27.615856	0.523140	-3.249609	15.787755	3.217552	7.969015
-1.732050808	-21.979819	8.242432	1.732051	83.082489	27.694163	0.524882	-3.299887	15.703407	3.299887	8.242432
-1.741341639	-22.302003	8.237465	1.746309	83.148794	27.716265	0.525374	-3.314481	15.679242	3.323936	8.322978
-1.771714948	-23.371524	8.220883	1.793264	83.360108	27.786703	0.526944	-3.362248	15.601068	3.403142	8.590358
-1.828427125	-25.435592	8.188479	1.882380	83.733278	27.911093	0.529721	-3.451682	15.458108	3.553534	9.106375
-1.9196940611	-28.943165	8.132133	2.029993	84.279900	28.093300	0.533800	-3.596280	15.234420	3.802909	9.983269
-2.00	-32.222252	8.077902	2.164530	84.711399	28.237133	0.537030	-3.724187	15.041807	4.030558	10.803040
-2.10	-36.562649	8.003595	2.338838	85.191744	28.397248	0.540636	-3.884315	14.804038	4.326086	11.888140
-2.20	-41.193895	7.920799	2.521633	85.616727	28.538909	0.543835	-4.045342	14.564701	4.636759	13.045951
-2.267949192	-44.509805	7.859038	2.651344	85.877681	28.625894	0.545804	-4.155244	14.399008	4.857684	13.874929
-2.747477419	-71.968876	7.227006	3.762904	87.239524	29.079841	0.556132	-4.940332	12.995124	6.766205	20.739696
-2.886751345	-81.335465	6.920135	4.209049	87.519210	29.173070	0.558264	-5.170939	12.395800	7.539526	23.081344
-3.100131380471	-96.963404	5.671282	5.671282	87.877987	29.292662	0.561005	-5.526027	10.109139	10.109139	26.988329
-3.732050808	-152.839404	IMAG.	IMAG.	88.601364	29.533788	0.566552	-6.587311	IMAG.	IMAG.	40.957328
-3.8284271247	-162.688068	IMAG.	IMAG.	88.680647	29.560216	0.567161	-6.750160	IMAG.	IMAG.	43.419494
-3.9953558031	-180.616651	IMAG.	IMAG.	88.804061	29.601354	0.568110	-7.032711	IMAG.	IMAG.	47.901640
-5	-313.313329	IMAG.	IMAG.	89.293342	29.764447	0.571882	-8.743068	IMAG.	IMAG.	81.075810
-5.67128182	-427.751550	IMAG.	IMAG.	89.477650	29.825883	0.573305	-9.892251	IMAG.	IMAG.	109.685365
-5.732050808	-439.207666	IMAG.	IMAG.	89.490941	29.830314	0.573408	-9.996457	IMAG.	IMAG.	112.549394
-5.846259147	-461.247550	IMAG.	IMAG.	89.514698	29.838233	0.573592	-10.192367	IMAG.	IMAG.	118.059365
-10	-1791.495748	IMAG.	IMAG.	89.872852	29.957617	0.576364	-17.350135	IMAG.	IMAG.	450.621414
-15	-5181.799780	IMAG.	IMAG.	89.955865	29.985288	0.577008	-25.996175	IMAG.	IMAG.	1298.197423
-17.16933693	-7436.781623	IMAG.	IMAG.	89.969228	29.989743	0.577112	-29.750463	IMAG.	IMAG.	1861.942883
-17.18882278704	-7459.493505	IMAG.	IMAG.	89.969322	29.989774	0.577112	-29.784190	IMAG.	IMAG.	1867.620854

FOR 3θ= 60°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	3493.467546	IMAG.	IMAG.	-89.934266	-29.978089	-0.576840	-29.798225	IMAG.	IMAG.	-871.634836
17.16933693	3479.752772	IMAG.	IMAG.	-89.934007	-29.978002	-0.576838	-29.764549	IMAG.	IMAG.	-868.206142
14	1685.286176	IMAG.	IMAG.	-89.863448	-29.954483	-0.576292	-24.293260	IMAG.	IMAG.	-419.589493
11.45945741782	789.845656	IMAG.	IMAG.	-89.707273	-29.902424	-0.575082	-19.926656	IMAG.	IMAG.	-195.729363
11.4300523	781.876573	IMAG.	IMAG.	-89.704263	-29.901421	-0.575059	-19.876329	IMAG.	IMAG.	-193.737092
10	452.116809	IMAG.	IMAG.	-89.485214	-29.828405	-0.573364	-17.440929	IMAG.	IMAG.	-111.297151
8.595086218	227.044750	IMAG.	IMAG.	-88.958925	-29.652975	-0.569303	-15.097569	IMAG.	IMAG.	-55.029137
8.555546781	221.964278	IMAG.	IMAG.	-88.934334	-29.644778	-0.569113	-15.033118	IMAG.	IMAG.	-53.759019
7	69.120582	IMAG.	IMAG.	-86.320001	-28.773334	-0.549149	-12.747002	IMAG.	IMAG.	-15.548095
5.8462591466	6.413106	IMAG.	IMAG.	7.337842	2.445947	0.042716	136.864136	IMAG.	IMAG.	0.128774
5.732050809	2.143594	-0.267949	-0.267949	50.103909	16.701303	0.300039	19.104342	-0.893047	-0.893047	1.196152
5.70	1.002059	0.185723	-0.689570	55.981631	18.660544	0.337714	16.878189	0.549941	-2.041877	1.481536
5.671281820	0.000000	0.363970	-0.839100	**60**	**20**	**0.363970**	15.581719	1.000000	-2.305407	**1.732051**
5.50	-5.576560	1.010797	-1.314644	72.261664	24.087221	0.447054	12.302765	2.261017	-2.940684	3.126191
5.00	-18.171760	2.095670	-1.899518	80.945325	26.981775	0.509125	9.820774	4.116220	-3.730947	6.274991
4.3032012473	-27.712813	3.100131	-2.207180	83.413224	27.804408	0.527339	8.160226	5.878826	-4.185509	8.660254
4.219331772	-28.316049	3.201356	-2.224536	83.524997	27.841666	0.528170	7.988588	6.061224	-4.211780	8.811063
3.8284271247	-29.799781	3.634101	-2.266375	83.784484	27.928161	0.530102	7.222055	6.855472	-4.275356	9.181996
3.732050808	-29.856406	3.732051	-2.267949	83.793977	27.931326	0.530173	7.039308	7.039308	-4.277753	9.196152
3.44991420939	-29.401258	4.001491	-2.255253	83.716844	27.905615	0.529598	6.514210	7.555710	-4.258422	9.082365
3.340232616	-28.995430	4.099760	-2.243840	83.646440	27.882147	0.529074	6.313358	7.748936	-4.241071	8.980908
3.141756715	-27.971410	4.269041	-2.214645	83.461607	27.820536	0.527698	5.953699	8.089927	-4.196802	8.724903
3.017830135	-27.160071	4.369417	-2.191095	83.307381	27.769127	0.526552	5.731308	8.298172	-4.161214	8.522069
2.886751345	-26.173212	4.471325	-2.161924	83.109743	27.703248	0.525084	5.497694	8.515447	-4.117292	8.275354
2.747477419	-24.994528	4.574962	-2.126287	82.857916	27.619305	0.523216	5.251129	8.743917	-4.063875	7.980683
2.267949192	-20.133284	4.896974	-1.968770	81.591904	27.197301	0.513871	4.413463	9.529584	-3.831257	6.765372
2.00	-17.052559	5.054397	-1.858244	80.530224	26.843408	0.506088	3.951885	9.987197	-3.671784	5.995191
1.8284271247	-15.012027	5.146959	-1.779233	79.667685	26.555895	0.499800	3.658316	10.298031	-3.559889	5.485058
1.732050808	-13.856406	5.196152	-1.732051	79.106605	26.368868	0.495727	3.493959	10.481878	-3.493959	5.196152
1.50	-11.084292	5.306344	-1.610191	77.479609	25.826536	0.483990	3.099235	10.963738	-3.326908	4.503124
1.40	-9.908408	5.350221	-1.554069	76.635578	25.545193	0.477944	2.929213	11.194241	-3.251571	4.209153

FOR 3θ= 60°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-8.752447	5.391911	-1.495759	75.689528	25.229843	0.471201	2.758910	11.442920	-3.174356	3.920163
1.191753595	-7.530564	5.434554	-1.430155	74.535961	24.845320	0.463025	2.573842	11.737062	-3.088721	3.614692
1.10	-6.524294	5.468659	-1.372506	73.440545	24.480182	0.455309	2.415944	12.010884	-3.014453	3.363124
1.00	-5.464102	5.503671	-1.307519	72.110909	24.036970	0.446002	2.242142	12.340013	-2.931643	3.098076
0.90	-4.447833	5.536404	-1.240252	70.627493	23.542498	0.435695	2.065667	12.707076	-2.846609	2.844009
0.839099631	-3.852998	5.555208	-1.198155	69.644331	23.214777	0.428906	1.956372	12.952045	-2.793516	2.695300
0.657710346	-2.204333	5.606038	-1.067595	66.346848	22.115616	0.406375	1.618480	13.795218	-2.627116	2.283134
0.502219976	-0.958536	5.643273	-0.949340	63.106767	21.035589	0.384577	1.305903	14.673979	-2.468532	1.971685
0.466307658	-0.695343	5.651017	-0.921173	62.314444	20.771481	0.379295	1.229407	14.898743	-2.428645	1.905887
0.431357893	-0.448606	5.658240	-0.893446	61.531687	20.510562	0.374095	1.153071	15.125151	-2.388287	1.844202
0.363970234	0.000000	5.671282	-0.839100	**60**	**20**	**0.363970**	1.000000	15.581719	-2.305407	**1.732051**
0.299380347	0.395019	5.682670	-0.785898	58.522578	19.507526	0.354266	0.845071	16.040668	-2.218381	1.633296
0.267949192	0.574374	5.687811	-0.759608	57.807876	19.269292	0.349593	0.766459	16.269788	-2.172833	1.588457
0.237004353	0.742477	5.692614	-0.733466	57.111270	19.037090	0.345052	0.686866	16.497851	-2.125669	1.546432
0.20648339	0.899864	5.697097	-0.707428	56.434426	18.811475	0.340651	0.606143	16.724131	-2.076692	1.507085
0.17632698	1.046997	5.701275	-0.681449	55.779102	18.593034	0.336402	0.524156	16.947812	-2.025701	1.470301
0.00	1.732051	**5.720575**	**-0.524423**	52.410911	17.470304	0.314729	0.000000	18.176190	-1.666267	1.299038
-0.10	1.979089	5.727474	-0.431322	51.053973	17.017991	0.306074	-0.326718	18.712707	-1.409207	1.237278
-0.17632698	2.093995	5.730672	-0.358193	50.394406	16.798135	0.301882	-0.584092	18.983136	-1.186532	1.208552
-0.267949192	2.143594	5.732051	-0.267949	50.103909	16.701303	0.300039	-0.893047	19.104342	-0.893047	1.196152
-0.363970234	2.087388	5.730489	-0.170366	50.432836	16.810945	0.302126	-1.204696	18.967199	-0.563890	1.210204
-0.40	2.036666	5.729078	-0.132925	50.725797	16.908599	0.303987	-1.315845	18.846445	-0.437273	1.222884
-0.502218876	1.801443	5.722516	-0.024145	52.037941	17.345980	0.312346	-1.607893	18.321087	-0.077302	1.281690
-0.657710346	1.172901	5.704841	0.149022	55.200264	18.400088	0.332657	-1.977140	17.149297	0.447974	1.438826
-0.700207538	0.941744	5.698287	0.198073	56.250162	18.750054	0.339455	-2.062739	16.786561	0.583502	1.496615
-0.744472416	0.672941	5.690629	0.249995	57.402646	19.134215	0.346950	-2.145763	16.401873	0.720552	1.563816
-0.839099631	0.000000	5.671282	0.363970	**60**	**20**	**0.363970**	-2.305407	15.581719	1.000000	**1.732051**
-0.943451341	-0.902459	5.644926	0.494677	62.941488	20.980496	0.383474	-2.460278	14.720512	1.289991	1.957666
-1.00	-1.464102	5.628279	0.567874	64.516265	21.505422	0.394020	-2.537944	14.284253	1.441232	2.098076
-1.059938076	-2.116657	5.608690	0.647400	66.143063	22.047688	0.404995	-2.617165	13.848798	1.598540	2.261215
-1.12369091	-2.876824	5.585526	0.734318	67.806801	22.602267	0.416306	-2.699193	13.416867	1.763888	2.451257

FOR 3θ= 60°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-3.765282	5.557959	0.829947	69.491211	23.163737	0.427852	-2.785437	12.990391	1.939800	2.673371
-1.20	-3.878409	5.554410	0.841743	69.688281	23.229427	0.429209	-2.795843	12.941049	1.961150	2.701653
-1.30	-5.346447	5.507501	0.988651	71.950512	23.983504	0.444884	-2.922112	12.379641	2.222269	3.068663
-1.40	-6.996408	5.452767	1.143385	73.972697	24.657566	0.459052	-3.049766	11.878331	2.490755	3.481153
-1.60	-10.866099	5.314604	1.481549	77.331012	25.777004	0.482924	-3.313152	11.005056	3.067872	4.448576
-1.70	-13.097830	5.227337	1.668815	78.704392	26.234797	0.492816	-3.449566	10.607086	3.386287	5.006508
-1.732050808	-13.856406	5.196152	1.732051	79.106605	26.368868	0.495727	-3.493959	10.481878	3.493959	5.196152
-1.741341639	-14.080284	5.186786	1.750708	79.219957	26.406652	0.496549	-3.506888	10.445668	3.525750	5.252122
-1.771714948	-14.824758	5.155077	1.812791	79.580675	26.526892	0.499168	-3.549338	10.327343	3.631626	5.438240
-1.828427125	-16.266861	5.090975	1.933604	80.215552	26.738517	0.503790	-3.629342	10.105348	3.838114	5.798766
-1.9196940611	-18.732364	4.971631	2.144216	81.139969	27.046656	0.510552	-3.760039	9.737764	4.199802	6.415142
-2.00	-21.052559	4.844610	2.351542	81.864383	27.288128	0.515876	-3.876900	9.391035	4.558346	6.995191
-2.10	-24.143981	4.641323	2.654829	82.664515	27.554838	0.521784	-4.024653	8.895103	5.087983	7.768046
-2.20	-27.465327	4.333869	3.062284	83.366249	27.788750	0.526989	-4.174658	8.223828	5.810903	8.598383
-2.267949192	-29.856406	3.732051	3.732051	83.793977	27.931326	0.530173	-4.277753	7.039308	7.039308	9.196152
-2.747477419	-49.989057	IMAG.	IMAG.	85.980008	28.660003	0.546577	-5.026698	IMAG.	IMAG.	14.229315
-2.886751345	-56.965227	IMAG.	IMAG.	86.417716	28.805905	0.549889	-5.249699	IMAG.	IMAG.	15.973357
-3.100131380471	-68.701600	IMAG.	IMAG.	86.972493	28.990831	0.554100	-5.594896	IMAG.	IMAG.	18.907451
-3.732050808	-111.425626	IMAG.	IMAG.	88.064313	29.354771	0.562431	-6.635565	IMAG.	IMAG.	29.588457
-3.8284271247	-119.054615	IMAG.	IMAG.	88.181449	29.393816	0.563329	-6.796078	IMAG.	IMAG.	31.495705
-3.9953558031	-133.004714	IMAG.	IMAG.	88.362639	29.454213	0.564718	-7.074953	IMAG.	IMAG.	34.983229
-5	-238.171760	IMAG.	IMAG.	89.065023	29.688341	0.570120	-8.770079	IMAG.	IMAG.	61.274991
-5.67128182	-330.788146	IMAG.	IMAG.	89.321406	29.773802	0.572098	-9.913124	IMAG.	IMAG.	84.429087
-5.732050808	-340.133284	IMAG.	IMAG.	89.339676	29.779892	0.572239	-10.016875	IMAG.	IMAG.	86.765372
-5.846259147	-358.144952	IMAG.	IMAG.	89.372252	29.790751	0.572491	-10.211966	IMAG.	IMAG.	91.268289
-10	-1487.883191	IMAG.	IMAG.	89.846681	29.948894	0.576162	-17.356242	IMAG.	IMAG.	373.702849
-15	-4497.402244	IMAG.	IMAG.	89.949119	29.983040	0.576956	-25.998532	IMAG.	IMAG.	1126.082612
-17.16933693	-6539.796004	IMAG.	IMAG.	89.964993	29.988331	0.577079	-29.752156	IMAG.	IMAG.	1636.681052
-17.18882278704	-6560.468407	IMAG.	IMAG.	89.965103	29.988368	0.577080	-29.785878	IMAG.	IMAG.	1641.849153

FOR 3θ= 50°		OTHER TWO ROOTS		θ’ FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	3971.828809	IMAG.	IMAG.	-89.942229	-29.980743	-0.576902	-29.795037	IMAG.	IMAG.	-991.765449
17.16933693	3957.028851	IMAG.	IMAG.	-89.942012	-29.980671	-0.576901	-29.761346	IMAG.	IMAG.	-988.065459
14	2002.440641	IMAG.	IMAG.	-89.885275	-29.961758	-0.576461	-24.286131	IMAG.	IMAG.	-499.418407
11.45945741782	1002.159495	IMAG.	IMAG.	-89.770219	-29.923406	-0.575569	-19.909781	IMAG.	IMAG.	-249.348120
11.4300523	993.099441	IMAG.	IMAG.	-89.768113	-29.922704	-0.575553	-19.859255	IMAG.	IMAG.	-247.083107
10	613.665676	IMAG.	IMAG.	-89.623616	-29.874539	-0.574434	-17.408430	IMAG.	IMAG.	-152.224665
8.595086218	346.248645	IMAG.	IMAG.	-89.328887	-29.776296	-0.572156	-15.022274	IMAG.	IMAG.	-85.370408
8.555546781	340.069004	IMAG.	IMAG.	-89.316520	-29.772173	-0.572061	-14.955665	IMAG.	IMAG.	-83.825497
7	148.003975	IMAG.	IMAG.	-88.400388	-29.466796	-0.565008	-12.389205	IMAG.	IMAG.	-35.809240
5.8462591466	61.272853	IMAG.	IMAG.	-85.950835	-28.650278	-0.546357	-10.700445	IMAG.	IMAG.	-14.126460
5.732050809	54.859971	IMAG.	IMAG.	-85.434531	-28.478177	-0.542463	-10.566720	IMAG.	IMAG.	-12.523239
5.70	53.124531	IMAG.	IMAG.	-85.271417	-28.423806	-0.541235	-10.531469	IMAG.	IMAG.	-12.089379
5.671281820	51.593150	IMAG.	IMAG.	-85.117511	-28.372504	-0.540078	-10.500858	IMAG.	IMAG.	-11.706534
5.50	42.915115	IMAG.	IMAG.	-84.014153	-28.004718	-0.531815	-10.341941	IMAG.	IMAG.	-9.537025
5.00	21.810234	IMAG.	IMAG.	-76.791878	-25.597293	-0.479062	-10.437071	IMAG.	IMAG.	-4.260805
4.3032012473	1.761817	-0.363970	-0.363970	36.917511	12.305837	0.218142	19.726607	-1.668502	-1.668502	0.751299
4.219331772	0.000000	0.299380	-0.943451	**50**	**16 2/3**	**0.299380**	14.093550	1.000000	-3.151347	**1.191754**
3.8284271247	-6.582906	1.304079	-1.557245	70.586248	23.528749	0.435409	8.792712	2.995066	-3.576511	2.837480
3.732050808	-7.820595	1.477560	-1.634350	72.371159	24.123720	0.447819	8.333846	3.299463	-3.649582	3.146902
3.44991420939	-10.649852	1.916416	-1.791070	75.454978	25.151659	0.469534	7.347525	4.081527	-3.814567	3.854217
3.340232616	-11.451190	2.066580	-1.831552	76.145281	25.381760	0.474445	7.040297	4.355786	-3.860411	4.054551
3.141756715	-12.512477	2.316501	-1.882997	76.966273	25.655424	0.480310	6.541107	4.822932	-3.920381	4.319873
3.017830135	-12.938424	2.460438	-1.903007	77.269493	25.756498	0.482483	6.254797	5.099538	-3.944199	4.426360
2.886751345	-13.206079	2.603915	-1.915406	77.453030	25.817677	0.483800	5.966833	5.382220	-3.959090	4.493273
2.747477419	-13.299311	2.747477	-1.919694	77.515743	25.838581	0.484250	5.673677	5.673677	-3.964263	4.516581
2.267949192	-12.336373	3.181932	-1.874621	76.836723	25.612241	0.479382	4.730981	6.637566	-3.910491	4.275847
2.00	-11.109290	3.389722	-1.814462	75.858768	25.286256	0.472404	4.233661	7.175468	-3.840908	3.969076
1.8284271247	-10.133447	3.510994	-1.764161	74.973335	24.991112	0.466119	3.922663	7.532402	-3.784788	3.725115
1.732050808	-9.534029	3.575261	-1.732051	74.373700	24.791233	0.461879	3.750008	7.740683	-3.750008	3.575261
1.50	-7.977583	3.718967	-1.643707	72.575093	24.191698	0.449244	3.338945	8.278286	-3.658831	3.186149
1.40	-7.271758	3.776181	-1.600920	71.620427	23.873476	0.442585	3.163232	8.532098	-3.617202	3.009693

FOR 3θ= 50°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-6.553437	3.830592	-1.555331	70.539505	23.513168	0.435086	2.987917	8.804224	-3.574770	2.830113
1.191753595	-5.768747	3.886346	-1.502839	69.210267	23.070089	0.425919	2.798074	9.124608	-3.528460	2.633940
1.10	-5.103312	3.931051	-1.455790	67.939497	22.646499	0.417212	2.636547	9.422184	-3.489327	2.467582
1.00	-4.383507	3.977101	-1.401840	66.388247	22.129416	0.406656	2.459080	9.780012	-3.447238	2.287630
0.90	-3.675208	4.020351	-1.345090	64.647989	21.549330	0.394905	2.279027	10.180543	-3.406108	2.110556
0.839099631	-3.252044	4.045310	-1.309149	63.489443	21.163148	0.387135	2.167462	10.449361	-3.381636	2.004765
0.657710346	-2.043460	4.113380	-1.195830	59.572980	19.857660	0.361159	1.821108	11.389378	-3.311087	1.702619
0.502219976	-1.090004	4.164112	-1.091071	55.669215	18.556405	0.335690	1.496081	12.404622	-3.250231	1.464255
0.466307658	-0.883189	4.174806	-1.065853	54.703830	18.234610	0.329453	1.415401	12.671941	-3.235222	1.412551
0.431357893	-0.687305	4.184838	-1.040936	53.745029	17.915010	0.323281	1.334315	12.944914	-3.219914	1.363580
0.363970234	-0.325571	4.203127	-0.991836	51.851935	17.283978	0.311159	1.169726	13.507990	-3.187559	1.273146
0.299380347	0.000000	4.219332	-0.943451	**50**	**16 2/3**	**0.299380**	1.000000	14.093550	-3.151347	**1.191754**
0.267949192	0.150452	4.226741	-0.919430	49.092854	16.364285	0.293639	0.912513	14.394356	-3.131159	1.154141
0.237004353	0.293227	4.233728	-0.895471	48.200195	16.066732	0.288006	0.822914	14.700124	-3.109208	1.118447
0.20648339	0.428674	4.240316	-0.871538	47.323584	15.774528	0.282491	0.730937	15.010425	-3.085185	1.084585
0.17632698	0.557096	4.246526	-0.847592	46.464671	15.488224	0.277103	0.636322	15.324709	-3.058761	1.052480
0.00	1.191754	**4.276731**	**-0.701470**	41.790829	13.930276	0.248036	0.000000	17.242392	-2.828101	0.893815
-0.10	1.455001	4.289030	-0.613769	39.624870	13.208290	0.234701	-0.426075	18.274478	-2.615116	0.828003
-0.17632698	1.604093	4.295938	-0.544350	38.334490	12.778163	0.226794	-0.777478	18.942053	-2.400199	0.790730
-0.267949192	1.719671	4.301265	-0.458055	37.301449	12.433816	0.220483	-1.215282	19.508367	-2.077506	0.761836
-0.363970234	1.761817	4.303201	-0.363970	36.917511	12.305837	0.218142	-1.668502	19.726607	-1.668502	0.751299
-0.40	1.755712	4.302921	-0.327660	36.973371	12.324457	0.218482	-1.830811	19.694584	-1.499709	0.752826
-0.502218876	1.669973	4.298977	-0.221498	37.749200	12.583067	0.223216	-2.249921	19.259253	-0.992300	0.774260
-0.657710346	1.333774	4.283382	-0.050411	40.639828	13.546609	0.240939	-2.729776	17.777849	-0.209228	0.858310
-0.700207538	1.196154	4.276938	-0.001469	41.755768	13.918589	0.247819	-2.825476	17.258290	-0.005930	0.892715
-0.744472416	1.031005	4.269156	0.050577	43.045542	14.348514	0.255799	-2.910383	16.689511	0.197723	0.934002
-0.839099631	0.600953	4.248640	0.165721	46.164978	15.388326	0.275227	-3.048758	15.436873	0.602125	1.041515
-0.943451341	0.000000	4.219332	0.299380	**50**	**16 2/3**	**0.299380**	-3.151347	14.093550	1.000000	**1.191754**
-1.00	-0.383507	4.200218	0.375043	52.166358	17.388786	0.313166	-3.193194	13.412111	1.197584	1.287630
-1.059938076	-0.835933	4.177235	0.457964	54.476581	18.158860	0.327988	-3.231639	12.735945	1.396284	1.400737
-1.12369091	-1.370452	4.149439	0.549513	56.906316	18.968772	0.343718	-3.269223	12.072217	1.598731	1.534367

FOR 3θ= 50°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-2.003465	4.115558	0.651456	59.425400	19.808467	0.360189	-3.308689	11.426104	1.808651	1.692620
-1.20	-2.084622	4.111134	0.664126	59.723525	19.907842	0.362150	-3.313546	11.352029	1.833845	1.712909
-1.30	-3.147437	4.051385	0.823876	63.187757	21.062586	0.385118	-3.375590	10.519857	2.139283	1.978613
-1.40	-4.359758	3.978583	0.996678	66.333552	22.111184	0.406285	-3.445854	9.792585	2.453147	2.281693
-1.60	-7.256914	3.777342	1.397919	71.599265	23.866422	0.442438	-3.616326	8.537561	3.159581	3.005982
-1.70	-8.953750	3.632191	1.643070	73.747068	24.582356	0.457463	-3.716145	7.939852	3.591698	3.430191
-1.732050808	-9.534029	3.575261	1.732051	74.373700	24.791233	0.461879	-3.750008	7.740683	3.750008	3.575261
-1.741341639	-9.705599	3.557492	1.759110	74.550048	24.850016	0.463125	-3.759985	7.681501	3.798352	3.618153
-1.771714948	-10.277119	3.494597	1.852379	75.110437	25.036812	0.467090	-3.793090	7.481634	3.965784	3.761033
-1.828427125	-11.388281	3.349695	2.053993	76.093422	25.364474	0.474075	-3.856829	7.065745	4.332630	4.038824
-1.9196940611	-13.299311	2.747477	2.747477	77.515743	25.838581	0.484250	-3.964263	5.673677	5.673677	4.516581
-2.0000000000	-15.109290	IMAG.	IMAG.	78.621513	26.207171	0.492216	-4.063253	IMAG.	IMAG.	4.969076
-2.10	-17.536146	IMAG.	IMAG.	79.832283	26.610761	0.500998	-4.191637	IMAG.	IMAG.	5.575790
-2.20	-20.160509	IMAG.	IMAG.	80.883736	26.961245	0.508674	-4.324973	IMAG.	IMAG.	6.231881
-2.267949192	-22.059496	IMAG.	IMAG.	81.519323	27.173108	0.513337	-4.418051	IMAG.	IMAG.	6.706628
-2.747477419	-38.293839	IMAG.	IMAG.	84.692922	28.230974	0.536892	-5.117379	IMAG.	IMAG.	10.765213
-2.886751345	-43.998093	IMAG.	IMAG.	85.310762	28.436921	0.541531	-5.330722	IMAG.	IMAG.	12.191277
-3.100131380471	-53.663809	IMAG.	IMAG.	86.083812	28.694604	0.547362	-5.663772	IMAG.	IMAG.	14.607706
-3.732050808	-89.389815	IMAG.	IMAG.	87.567405	29.189135	0.558632	-6.680693	IMAG.	IMAG.	23.539207
-3.8284271247	-95.837740	IMAG.	IMAG.	87.723145	29.241048	0.559822	-6.838655	IMAG.	IMAG.	25.151188
-3.9953558031	-107.670932	IMAG.	IMAG.	87.962552	29.320851	0.561652	-7.113573	IMAG.	IMAG.	28.109487
-5	-198.189766	IMAG.	IMAG.	88.870925	29.623642	0.568625	-8.793142	IMAG.	IMAG.	50.739195
-5.67128182	-279.194996	IMAG.	IMAG.	89.192963	29.730988	0.571107	-9.930332	IMAG.	IMAG.	70.990503
-5.732050808	-287.416907	IMAG.	IMAG.	89.215669	29.738556	0.571282	-10.033660	IMAG.	IMAG.	73.045980
-5.846259147	-303.285206	IMAG.	IMAG.	89.256067	29.752022	0.571594	-10.227994	IMAG.	IMAG.	77.013055
-10	-1326.334324	IMAG.	IMAG.	89.827825	29.942608	0.576015	-17.360645	IMAG.	IMAG.	332.775335
-15	-4133.241922	IMAG.	IMAG.	89.944615	29.981538	0.576921	-26.000106	IMAG.	IMAG.	1034.502234
-17.16933693	-6062.519925	IMAG.	IMAG.	89.962226	29.987409	0.577057	-29.753262	IMAG.	IMAG.	1516.821735
-17.18882278704	-6082.107145	IMAG.	IMAG.	89.962348	29.987449	0.577058	-29.786981	IMAG.	IMAG.	1521.718540

FOR 3θ= 45°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	4141.601090	IMAG.	IMAG.	-89.944610	-29.981537	-0.576921	-29.794083	IMAG.	IMAG.	-1034.400273
17.16933693	4126.415996	IMAG.	IMAG.	-89.944406	-29.981469	-0.576919	-29.760389	IMAG.	IMAG.	-1030.603999
14	2115.000000	IMAG.	IMAG.	-89.891434	-29.963811	-0.576508	-24.284120	IMAG.	IMAG.	-527.750000
11.45945741782	1077.510506	IMAG.	IMAG.	-89.786512	-29.928837	-0.575695	-19.905417	IMAG.	IMAG.	-268.377627
11.4300523	1068.063262	IMAG.	IMAG.	-89.784616	-29.928205	-0.575681	-19.854846	IMAG.	IMAG.	-266.015815
10	671.000000	IMAG.	IMAG.	-89.656401	-29.885467	-0.574688	-17.400745	IMAG.	IMAG.	-166.750000
8.595086218	388.554573	IMAG.	IMAG.	-89.404051	-29.801350	-0.572737	-15.007046	IMAG.	IMAG.	-96.138643
8.555546781	381.984833	IMAG.	IMAG.	-89.393694	-29.797898	-0.572657	-14.940098	IMAG.	IMAG.	-94.496208
7	176.000000	IMAG.	IMAG.	-88.667780	-29.555927	-0.567062	-12.344328	IMAG.	IMAG.	-43.000000
5.8462591466	80.742791	IMAG.	IMAG.	-87.016320	-29.005440	-0.554433	-10.544569	IMAG.	IMAG.	-19.185698
5.732050809	73.569219	IMAG.	IMAG.	-86.709305	-28.903102	-0.552100	-10.382264	IMAG.	IMAG.	-17.392305
5.70	71.623000	IMAG.	IMAG.	-86.614815	-28.871605	-0.551383	-10.337638	IMAG.	IMAG.	-16.905750
5.671281820	69.903760	IMAG.	IMAG.	-86.526718	-28.842239	-0.550715	-10.298033	IMAG.	IMAG.	-16.475940
5.50	60.125000	IMAG.	IMAG.	-85.923452	-28.641151	-0.546150	-10.070498	IMAG.	IMAG.	-14.031250
5.00	36.000000	IMAG.	IMAG.	-82.874984	-27.624995	-0.523343	-9.553964	IMAG.	IMAG.	-8.000000
4.3032012473	12.222479	IMAG.	IMAG.	-64.058408	-21.352803	-0.390946	-11.007157	IMAG.	IMAG.	-2.055620
4.219331772	10.049476	IMAG.	IMAG.	-56.526754	-18.842251	-0.341251	-12.364311	IMAG.	IMAG.	-1.512369
3.8284271247	1.656854	-0.414214	-0.414214	30.361193	10.120398	0.178494	21.448437	-2.320596	-2.320596	0.585786
3.732050808	0.000000	0.267949	-1.000000	**45**	**15**	**0.267949**	13.928203	1.000000	-3.732051	**1.000000**
3.44991420939	-3.994905	0.999151	-1.449065	63.420346	21.140115	0.386672	8.922058	2.583972	-3.747526	1.998726
3.340232616	-5.224670	1.205559	-1.545792	66.557488	22.185829	0.407804	8.190780	2.956223	-3.790527	2.306168
3.141756715	-7.026041	1.529015	-1.670772	70.060366	23.353455	0.431775	7.276383	3.541235	-3.869547	2.756510
3.017830135	-7.891106	1.707555	-1.725386	71.407788	23.802596	0.441107	6.841497	3.871072	-3.911493	2.972777
2.886751345	-8.603993	1.881488	-1.768239	72.392657	24.130886	0.447969	6.444092	4.200043	-3.947238	3.150998
2.747477419	-9.148632	2.052330	-1.799808	73.079497	24.359832	0.452775	6.068085	4.532781	-3.975059	3.287158
2.267949192	-9.569219	2.555598	-1.823547	73.575283	24.525094	0.456255	4.970790	5.601246	-3.996770	3.392305
2.00	-9.000000	2.791288	-1.791288	72.897271	24.299090	0.451498	4.429697	6.182279	-3.967431	3.250000
1.8284271247	-8.402020	2.927854	-1.756281	72.124030	24.041343	0.446094	4.098752	6.563318	-3.937024	3.100505
1.732050808	-8.000000	3.000000	-1.732051	71.565051	23.855017	0.442200	3.916894	6.784260	-3.916894	3.000000
1.50	-6.875000	3.160913	-1.660913	69.805666	23.268555	0.430018	3.488229	7.350659	-3.862430	2.718750
1.40	-6.336000	3.224871	-1.624871	68.843717	22.947906	0.423402	3.306549	7.616567	-3.837654	2.584000

FOR 3θ= 45°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-5.773000	3.285672	-1.585672	67.741161	22.580387	0.415858	3.126065	7.900943	-3.813012	2.443250
1.191753595	-5.143471	3.347977	-1.539731	66.372035	22.124012	0.406546	2.931410	8.235172	-3.787346	2.285868
1.10	-4.599000	3.397958	-1.497958	65.053547	21.684516	0.397635	2.766354	8.545414	-3.767166	2.149750
1.00	-4.000000	3.449490	-1.449490	63.434949	21.144983	0.386770	2.585515	8.918707	-3.747677	2.000000
0.90	-3.401000	3.497958	-1.397958	61.610219	20.536740	0.374616	2.402462	9.337457	-3.731713	1.850250
0.839099631	-3.038763	3.525973	-1.365073	60.391226	20.130409	0.366550	2.289182	9.619353	-3.724112	1.759691
0.657710346	-1.986365	3.602632	-1.260342	56.249745	18.749915	0.339453	1.937562	10.613062	-3.712866	1.496591
0.502219976	-1.136662	3.660147	-1.162367	52.091545	17.363848	0.312688	1.606137	11.705421	-3.717336	1.284166
0.466307658	-0.949856	3.672336	-1.138643	51.058174	17.019391	0.306101	1.523379	11.997145	-3.719831	1.237464
0.431357893	-0.772020	3.683797	-1.115155	50.029606	16.676535	0.299568	1.439933	12.297029	-3.722543	1.193005
0.363970234	-0.441117	3.704770	-1.068740	47.991449	15.997150	0.286692	1.269553	12.922494	-3.727840	1.110279
0.299380347	-0.140194	3.723462	-1.022842	45.986674	15.328891	0.274111	1.092186	13.583768	-3.731487	1.035048
0.267949192	0.000000	3.732051	-1.000000	**45**	**15**	**0.267949**	1.000000	13.928203	-3.732051	**1.000000**
0.237004353	0.133787	3.740179	-0.977184	44.025622	14.675207	0.261883	0.905002	14.281891	-3.731380	**0.966553**
0.20648339	0.261447	3.747875	-0.954358	43.064997	14.354999	0.255919	0.806830	14.644750	-3.729137	0.934638
0.17632698	0.383228	3.755162	-0.931488	42.119669	14.039890	0.250068	0.705117	15.016585	-3.724947	0.904193
0.00	1.000000	**3.791288**	**-0.791288**	36.869898	12.289966	0.217852	0.000000	17.403058	-3.632230	0.750000
-0.10	1.269000	3.806657	-0.706657	34.323307	11.441102	0.202382	-0.494115	18.809264	-3.491698	0.682750
-0.17632698	1.430225	3.815761	-0.639434	32.718462	10.906154	0.192681	-0.915124	19.803511	-3.318615	0.642444
-0.267949192	1.569219	3.823547	-0.555598	31.286847	10.428949	0.184057	-1.455797	20.773750	-3.018625	0.607695
-0.363970234	1.646271	3.827839	-0.463869	30.473928	10.157976	0.179171	-2.031409	21.364131	-2.588968	0.588432
-0.40	1.656000	3.828380	-0.428380	30.370303	10.123434	0.178549	-2.240280	21.441602	-2.399226	0.586000
-0.502218876	1.623314	3.826562	-0.324343	30.717578	10.239193	0.180635	-2.780302	21.183985	-1.795576	0.594172
-0.657710346	1.390868	3.813546	-0.155836	33.115726	11.038575	0.195079	-3.371506	19.548715	-0.798833	0.652283
-0.700207538	1.286446	3.807646	-0.107438	34.152521	11.384174	0.201348	-3.477600	18.910777	-0.533595	0.678389
-0.744472416	1.158084	3.800347	-0.055875	35.392995	11.797665	0.208868	-3.564314	18.194940	-0.267512	0.710479
-0.839099631	0.814234	3.780540	0.058560	38.535269	12.845090	0.228022	-3.679904	16.579704	0.256816	0.796441
-0.943451341	0.320286	3.751402	0.192049	42.611835	14.203945	0.253112	-3.727404	14.821105	0.758752	0.919928
-1.00	0.000000	3.732051	0.267949	**45**	**15**	**0.267949**	-3.732051	13.928203	1.000000	**1.000000**
-1.059938076	-0.381399	3.708507	0.351431	47.605472	15.868491	0.284263	-3.728723	13.046043	1.236287	1.095350
-1.12369091	-0.835835	3.679700	0.443991	50.403871	16.801290	0.301942	-3.721541	12.186762	1.470450	1.208959

FOR 3θ= 45°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-1.378189	3.644154	0.547600	53.360166	17.786722	0.320809	-3.714835	11.359254	1.706932	1.344547
-1.20	-1.448000	3.639480	0.560520	53.713349	17.904450	0.323077	-3.714285	11.265057	1.734941	1.362000
-1.30	-2.367000	3.575658	0.724342	57.861345	19.287115	0.349943	-3.714895	10.217841	2.069889	1.591750
-1.40	-3.424000	3.496148	0.903852	61.684519	20.561506	0.375109	-3.732251	9.320359	2.409573	1.856000
-1.60	-5.976000	3.264365	1.335635	68.151075	22.717025	0.418658	-3.821733	7.797207	3.190275	2.494000
-1.70	-7.483000	3.079726	1.620274	70.794677	23.598226	0.436852	-3.891475	7.049809	3.708974	2.870750
-1.732050808	-8.000000	3.000000	1.732051	71.565051	23.855017	0.442200	-3.916894	6.784260	3.916894	3.000000
-1.741341639	-8.153006	2.973816	1.767525	71.781729	23.927243	0.443708	-3.924522	6.702193	3.983533	3.038252
-1.771714948	-8.663144	2.873912	1.897803	72.469854	24.156618	0.448508	-3.950241	6.407716	4.231369	3.165786
-1.828427125	-9.656854	2.414214	2.414214	73.675050	24.558350	0.456957	-4.001313	5.283243	5.283243	3.414214
-1.9196940611	-11.371099	IMAG.	IMAG.	75.413514	25.137838	0.469240	-4.091073	IMAG.	IMAG.	3.842775
-2.00	-13.000000	IMAG.	IMAG.	76.759480	25.586493	0.478830	-4.176849	IMAG.	IMAG.	4.250000
-2.10	-15.191000	IMAG.	IMAG.	78.226346	26.075449	0.489364	-4.291287	IMAG.	IMAG.	4.797750
-2.20	-17.568000	IMAG.	IMAG.	79.493303	26.497768	0.498533	-4.412948	IMAG.	IMAG.	5.392000
-2.267949192	-19.292342	IMAG.	IMAG.	80.255630	26.751877	0.504083	-4.499162	IMAG.	IMAG.	5.823085
-2.747477419	-34.143161	IMAG.	IMAG.	84.013383	28.004461	0.531809	-5.166283	IMAG.	IMAG.	9.535790
-2.886751345	-39.396007	IMAG.	IMAG.	84.733678	28.244559	0.537197	-5.373729	IMAG.	IMAG.	10.849002
-3.100131380471	-48.326837	IMAG.	IMAG.	85.628662	28.542887	0.543925	-5.699554	IMAG.	IMAG.	13.081709
-3.732050808	-81.569219	IMAG.	IMAG.	87.323612	29.107871	0.556773	-6.703005	IMAG.	IMAG.	21.392305
-3.8284271247	-87.597980	IMAG.	IMAG.	87.499534	29.166511	0.558114	-6.859575	IMAG.	IMAG.	22.899495
-3.9953558031	-98.679874	IMAG.	IMAG.	87.769112	29.256371	0.560173	-7.132361	IMAG.	IMAG.	25.669968
-5	-184.000000	IMAG.	IMAG.	88.781125	29.593708	0.567934	-8.803843	IMAG.	IMAG.	47.000000
-5.67128182	-260.884385	IMAG.	IMAG.	89.134846	29.711615	0.570659	-9.938134	IMAG.	IMAG.	66.221096
-5.732050808	-268.707658	IMAG.	IMAG.	89.159662	29.719887	0.570850	-10.041255	IMAG.	IMAG.	68.176915
-5.846259147	-283.815267	IMAG.	IMAG.	89.211784	29.737261	0.571252	-10.234113	IMAG.	IMAG.	72.685868
-10	-1269.000000	IMAG.	IMAG.	89.820380	29.940127	0.575958	-17.362383	IMAG.	IMAG.	318.982051
-15	-4004.000000	IMAG.	IMAG.	89.942860	29.980953	0.576907	-26.000719	IMAG.	IMAG.	1002.732051
-17.16933693	-5893.132780	IMAG.	IMAG.	89.961156	29.987052	0.577049	-29.753690	IMAG.	IMAG.	1475.015246
-17.18882278704	-5912.334863	IMAG.	IMAG.	89.961282	29.987094	0.577050	-29.787408	IMAG.	IMAG.	1479.815767

FOR 3θ= 40°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	4284.056949	IMAG.	IMAG.	-89.946461	-29.982154	-0.576935	-29.793342	IMAG.	IMAG.	-1070.175138
17.16933693	4268.548687	IMAG.	IMAG.	-89.946267	-29.982089	-0.576934	-29.759645	IMAG.	IMAG.	-1066.298072
14	2209.448517	IMAG.	IMAG.	-89.896114	-29.965371	-0.576545	-24.282592	IMAG.	IMAG.	-551.523030
11.45945741782	1140.737512	IMAG.	IMAG.	-89.798500	-29.932833	-0.575788	-19.902207	IMAG.	IMAG.	-284.345278
11.4300523	1130.965376	IMAG.	IMAG.	-89.796754	-29.932251	-0.575775	-19.851604	IMAG.	IMAG.	-281.902244
10	719.109210	IMAG.	IMAG.	-89.679805	-29.893268	-0.574869	-17.395262	IMAG.	IMAG.	-178.938203
8.595086218	424.053462	IMAG.	IMAG.	-89.455246	-29.818415	-0.573132	-14.996688	IMAG.	IMAG.	-105.174266
8.555546781	417.156389	IMAG.	IMAG.	-89.446167	-29.815389	-0.573062	-14.929527	IMAG.	IMAG.	-103.449998
7	199.491454	IMAG.	IMAG.	-88.831665	-29.610555	-0.568323	-12.316945	IMAG.	IMAG.	-49.033764
5.8462591466	97.080009	IMAG.	IMAG.	-87.556175	-29.185392	-0.558547	-10.466915	IMAG.	IMAG.	-23.430903
5.732050809	89.268143	IMAG.	IMAG.	-87.334268	-29.111423	-0.556854	-10.293633	IMAG.	IMAG.	-21.477936
5.70	87.145059	IMAG.	IMAG.	-87.266823	-29.088941	-0.556340	-10.245532	IMAG.	IMAG.	-20.947165
5.671281820	85.268187	IMAG.	IMAG.	-87.204295	-29.068098	-0.555864	-10.202646	IMAG.	IMAG.	-20.477947
5.50	74.565808	IMAG.	IMAG.	-86.784940	-28.928313	-0.552675	-9.951607	IMAG.	IMAG.	-17.802352
5.00	47.906627	IMAG.	IMAG.	-84.869382	-28.289794	-0.538215	-9.289972	IMAG.	IMAG.	-11.137557
4.3032012473	21.000016	IMAG.	IMAG.	-77.226348	-25.742116	-0.482173	-8.924598	IMAG.	IMAG.	-4.410904
4.219331772	18.481988	IMAG.	IMAG.	-75.187088	-25.062363	-0.467633	-9.022733	IMAG.	IMAG.	-3.781397
3.8284271247	8.570834	IMAG.	IMAG.	-52.508140	-17.502713	-0.315351	-12.140215	IMAG.	IMAG.	-1.303609
3.7320508079	6.562259	IMAG.	IMAG.	-38.710956	-12.903652	-0.229098	-16.290218	IMAG.	IMAG.	-0.801465
3.44991420939	1.589259	-0.466308	-0.466308	23.835120	7.945040	0.139563	24.719442	-3.341203	-3.341203	0.441785
3.340232616	0.000000	0.237004	-1.059938	**40**	**13 1/3**	**0.237004**	14.093550	1.000000	-4.472230	**0.839100**
3.141756715	-2.422375	0.753413	-1.377871	55.309468	18.436489	0.333363	9.424426	2.260036	-4.133242	1.444693
3.017830135	-3.655903	0.995573	-1.496105	60.298436	20.099479	0.365938	8.246841	2.720609	-4.088413	1.753075
2.886751345	-4.742384	1.217988	-1.587440	63.715169	21.238390	0.388645	7.427724	3.133930	-4.084546	2.024696
2.747477419	-5.665800	1.427906	-1.658085	66.089851	22.029950	0.404634	6.790024	3.528880	-4.097735	2.255550
2.267949192	-7.247301	2.017043	-1.767693	69.332179	23.110726	0.426757	5.314375	4.726438	-4.142149	2.650925
2.00	-7.230096	2.283865	-1.766566	69.301435	23.100478	0.426546	4.688826	5.354322	-4.141560	2.646624
1.8284271247	-6.949180	2.436848	-1.747976	68.786811	22.928937	0.423012	4.322402	5.760709	-4.132216	2.576395
1.732050808	-6.712797	2.517299	-1.732051	68.334490	22.778163	0.419913	4.124786	5.994812	-4.124786	2.517299
1.50	-5.949823	2.696039	-1.678740	66.741000	22.247000	0.409050	3.667036	6.590982	-4.104000	2.326555
1.40	-5.550806	2.766886	-1.649587	65.816359	21.938786	0.402784	3.475809	6.869404	-4.095463	2.226801

FOR 3θ= 40°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-5.118135	2.834170	-1.616871	64.732579	21.577526	0.395474	3.287192	7.166508	-4.088435	2.118634
1.191753595	-4.618802	2.903083	-1.577538	63.363727	21.121242	0.386294	3.085096	7.515219	-4.083776	1.993800
1.10	-4.175832	2.958363	-1.541064	62.029410	20.676470	0.377399	2.914685	7.838816	-4.083380	1.883058
1.00	-3.678199	3.015383	-1.498084	60.376654	20.125551	0.366454	2.728857	8.228550	-4.088058	1.758649
0.90	-3.170912	3.069066	-1.451767	58.499626	19.499875	0.354116	2.541539	8.666835	-4.099692	1.631828
0.839099631	-2.859800	3.100131	-1.421932	57.239524	19.079841	0.345887	2.425935	8.962843	-4.110972	1.554050
0.657710346	-1.938458	3.185362	-1.325774	52.930770	17.643590	0.318056	2.067906	10.015092	-4.168363	1.323714
0.502219976	-1.175813	3.249670	-1.234591	48.569302	16.189767	0.290333	1.729806	11.192898	-4.252324	1.133053
0.466307658	-1.005797	3.263359	-1.212368	47.480126	15.826709	0.283475	1.644969	11.511982	-4.276808	1.090549
0.431357893	-0.843104	3.276258	-1.190317	46.393794	15.464598	0.276659	1.559167	11.842215	-4.302467	1.049876
0.363970234	-0.538072	3.299940	-1.146611	44.234142	14.744714	0.263179	1.382974	12.538747	-4.356766	0.973618
0.299380347	-0.257830	3.321151	-1.103232	42.099618	14.033206	0.249944	1.197791	13.287598	-4.413924	0.903557
0.267949192	-0.126244	3.330940	-1.081590	41.044806	13.681602	0.243434	1.100708	13.683157	-4.443061	0.870661
0.237004353	0.000000	3.340233	-1.059938	**40**	**13 1/3**	**0.237004**	1.000000	14.093550	-4.472230	**0.839100**
0.20648339	0.121127	3.349060	-1.038245	38.966552	12.988851	0.230663	0.895173	14.519264	-4.501129	0.808818
0.17632698	0.237335	3.357450	-1.016479	37.945889	12.648630	0.224418	0.785708	14.960713	-4.529402	0.779766
0.00	0.839100	**3.399724**	**-0.882425**	32.183222	10.727741	0.189453	0.000000	17.944901	-4.657739	0.629325
-0.10	1.112927	3.418349	-0.801050	29.286671	9.762224	0.172051	-0.581223	19.868224	-4.655887	0.560868
-0.17632698	1.284332	3.429827	-0.736201	27.384902	9.128301	0.160681	-1.097375	21.345604	-4.581762	0.518017
-0.267949192	1.442975	3.440329	-0.655081	25.564391	8.521464	0.149834	-1.788307	22.960939	-4.372046	0.478356
-0.363970234	1.549316	3.447306	-0.566037	24.312053	8.104018	0.142393	-2.556103	24.209867	-3.975187	0.451771
-0.40	1.572332	3.448810	-0.531511	24.037669	8.012556	0.140764	-2.841629	24.500598	-3.775893	0.446017
-0.502218876	1.584162	3.449582	-0.430064	23.896175	7.965392	0.139925	-3.589202	24.653091	-3.073534	0.443059
-0.657710346	1.438776	3.440053	-0.265043	25.613322	8.537774	0.150125	-4.381083	22.914578	-1.765484	0.479406
-0.700207538	1.362209	3.434997	-0.217490	26.498428	8.832809	0.155395	-4.505997	22.104995	-1.399599	0.498547
-0.744472416	1.264715	3.428520	-0.166749	27.606003	9.202001	0.162000	-4.595495	21.163639	-1.029310	0.522921
-0.839099631	0.993198	3.410250	-0.053852	30.574600	10.191533	0.179776	-4.667477	18.969456	-0.299548	0.590800
-0.943451341	0.589038	3.382389	0.078362	34.677036	11.559012	0.204525	-4.612888	16.537767	0.383139	0.691840
-1.00	0.321801	3.363501	0.153798	37.185752	12.395251	0.219777	-4.550059	15.304129	0.699789	0.758649
-1.059938076	0.000000	3.340233	0.237004	**40**	**13 1/3**	**0.237004**	-4.472230	14.093550	1.000000	**0.839100**
-1.12369091	-0.387237	3.311419	0.329571	43.103833	14.367944	0.256160	-4.386674	12.927146	1.286582	0.935909

FOR 3θ= 40°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-0.853520	3.275438	0.433614	46.464672	15.488224	0.277103	-4.300757	11.820282	1.564811	1.052480
-1.20	-0.913811	3.270676	0.446623	46.871337	15.623779	0.279652	-4.291041	11.695503	1.597064	1.067552
-1.30	-1.712135	3.204973	0.612326	51.720105	17.240035	0.310318	-4.189257	10.328042	1.973224	1.267134
-1.40	-2.638806	3.121443	0.795856	56.288786	18.762929	0.339706	-4.121212	9.188663	2.342778	1.498801
-1.60	-4.901186	2.865098	1.252201	64.154305	21.384768	0.391589	-4.085916	7.316593	3.197742	2.064396
-1.70	-6.248894	2.634952	1.582347	67.391345	22.463782	0.413473	-4.111512	6.372728	3.826964	2.401323
-1.732050808	-6.712797	2.517299	1.732051	68.334490	22.778163	0.419913	-4.124786	5.994812	4.124786	2.517299
-1.741341639	-6.850226	2.473205	1.785436	68.599672	22.866557	0.421729	-4.129055	5.864443	4.233610	2.551656
-1.771714948	-7.308858	2.144507	2.144507	69.441464	23.147155	0.427509	-4.144273	5.016282	5.016282	2.666314
-1.828427125	-8.204014	IMAG.	IMAG.	70.913950	23.637983	0.437679	-4.177553	IMAG.	IMAG.	2.890103
-1.9196940611	-9.753137	IMAG.	IMAG.	73.031930	24.343977	0.452442	-4.242965	IMAG.	IMAG.	3.277384
-2.00	-11.230096	IMAG.	IMAG.	74.664971	24.888324	0.463937	-4.310931	IMAG.	IMAG.	3.646624
-2.10	-13.223188	IMAG.	IMAG.	76.435997	25.478666	0.476519	-4.406964	IMAG.	IMAG.	4.144897
-2.20	-15.392627	IMAG.	IMAG.	77.956817	25.985606	0.487422	-4.513546	IMAG.	IMAG.	4.687256
-2.267949192	-16.970424	IMAG.	IMAG.	78.867335	26.289112	0.493994	-4.591043	IMAG.	IMAG.	5.081706
-2.747477419	-30.660328	IMAG.	IMAG.	83.293432	27.764477	0.526448	-5.218895	IMAG.	IMAG.	8.504182
-2.886751345	-35.534398	IMAG.	IMAG.	84.127658	28.042553	0.532662	-5.419476	IMAG.	IMAG.	9.722699
-3.100131380471	-43.848587	IMAG.	IMAG.	85.156509	28.385503	0.540371	-5.737042	IMAG.	IMAG.	11.801246
-3.732050808	-75.006961	IMAG.	IMAG.	87.077915	29.025972	0.554902	-6.725606	IMAG.	IMAG.	19.590840
-3.8284271247	-80.684000	IMAG.	IMAG.	87.274998	29.091666	0.556402	-6.880681	IMAG.	IMAG.	21.010100
-3.9953558031	-91.135480	IMAG.	IMAG.	87.576021	29.192007	0.558698	-7.151190	IMAG.	IMAG.	23.622970
-5	-172.093373	IMAG.	IMAG.	88.693966	29.564655	0.567263	-8.814248	IMAG.	IMAG.	43.862443
-5.67128182	-245.519959	IMAG.	IMAG.	89.079208	29.693069	0.570230	-9.945611	IMAG.	IMAG.	62.219089
-5.732050808	-253.008735	IMAG.	IMAG.	89.106101	29.702034	0.570437	-10.048526	IMAG.	IMAG.	64.091283
-5.846259147	-267.478049	IMAG.	IMAG.	89.153850	29.717950	0.570805	-10.242126	IMAG.	IMAG.	67.708612
-10	-1220.890790	IMAG.	IMAG.	89.812797	29.937599	0.575899	-17.364154	IMAG.	IMAG.	306.061797
-15	-3895.553151	IMAG.	IMAG.	89.941219	29.980406	0.576894	-26.001293	IMAG.	IMAG.	974.727387
-17.16933693	-5751.000089	IMAG.	IMAG.	89.960172	29.986724	0.577041	-29.754083	IMAG.	IMAG.	1438.589122
-17.18882278704	-5769.879004	IMAG.	IMAG.	89.960302	29.986767	0.577042	-29.787800	IMAG.	IMAG.	1443.308851

FOR 3θ=35°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	4407.027409	IMAG.	IMAG.	-89.947963	-29.982654	-0.576947	-29.792740	IMAG.	IMAG.	-1101.056645
17.16933693	4391.240183	IMAG.	IMAG.	-89.947776	-29.982592	-0.576945	-29.759041	IMAG.	IMAG.	-1097.109838
14	2290.978175	IMAG.	IMAG.	-89.899840	-29.966613	-0.576574	-24.281376	IMAG.	IMAG.	-572.044336
11.45945741782	1195.316201	IMAG.	IMAG.	-89.807816	-29.935939	-0.575860	-19.899712	IMAG.	IMAG.	-298.128843
11.4300523	1185.263613	IMAG.	IMAG.	-89.806182	-29.935394	-0.575848	-19.849086	IMAG.	IMAG.	-295.615696
10	760.637946	IMAG.	IMAG.	-89.697585	-29.899195	-0.575007	-17.391098	IMAG.	IMAG.	-189.459279
8.595086218	454.696741	IMAG.	IMAG.	-89.492854	-29.830951	-0.573423	-14.989085	IMAG.	IMAG.	-112.973978
8.555546781	447.517109	IMAG.	IMAG.	-89.484667	-29.828222	-0.573360	-14.921779	IMAG.	IMAG.	-111.179070
7	219.769699	IMAG.	IMAG.	-88.943825	-29.647942	-0.569186	-12.298257	IMAG.	IMAG.	-54.242217
5.8462591466	111.182590	IMAG.	IMAG.	-87.886368	-29.295456	-0.561070	-10.419847	IMAG.	IMAG.	-27.095440
5.732050809	102.819736	IMAG.	IMAG.	-87.709822	-29.236607	-0.559720	-10.240927	IMAG.	IMAG.	-25.004726
5.70	100.543979	IMAG.	IMAG.	-87.656559	-29.218853	-0.559313	-10.191073	IMAG.	IMAG.	-24.435787
5.671281820	98.531036	IMAG.	IMAG.	-87.607340	-29.202447	-0.558937	-10.146547	IMAG.	IMAG.	-23.932552
5.50	87.031373	IMAG.	IMAG.	-87.281140	-29.093713	-0.556449	-9.884102	IMAG.	IMAG.	-21.057636
5.00	58.184642	IMAG.	IMAG.	-85.869084	-28.623028	-0.545739	-9.161885	IMAG.	IMAG.	-13.845953
4.3032012473	28.576944	IMAG.	IMAG.	-81.179060	-27.059687	-0.510838	-8.423803	IMAG.	IMAG.	-6.444028
4.219331772	25.761084	IMAG.	IMAG.	-80.117451	-26.705817	-0.503075	-8.387086	IMAG.	IMAG.	-5.740063
3.8284271247	14.539105	IMAG.	IMAG.	-71.182657	-23.727552	-0.439543	-8.710018	IMAG.	IMAG.	-2.934569
3.732050808	12.226919	IMAG.	IMAG.	-67.005857	-22.335286	-0.410850	-9.083742	IMAG.	IMAG.	-2.356522
3.44991420939	6.409609	IMAG.	IMAG.	-42.056614	-14.018871	-0.249678	-13.817461	IMAG.	IMAG.	-0.902195
3.340232616	4.510029	IMAG.	IMAG.	-23.137008	-7.712336	-0.135425	-24.664897	IMAG.	IMAG.	-0.427300
3.141756715	1.551592	-0.520567	-0.520567	17.344078	5.781359	0.101248	31.030429	-5.141525	-5.141525	0.312309
3.017830135	0.000000	0.206483	-1.123691	**35**	**11 2/3**	**0.206483**	14.615365	1.000000	-5.442040	**0.700208**
2.886751345	-1.408974	0.547755	-1.333884	46.463891	15.487964	0.277098	10.417787	1.976754	-4.813757	1.052451
2.747477419	-2.659356	0.828695	-1.475550	53.774398	17.924799	0.323469	8.493782	2.561898	-4.561639	1.365046
2.267949192	-5.242980	1.537297	-1.704623	63.559908	21.186636	0.387606	5.851170	3.966131	-4.397823	2.010953
2.00	-5.702283	1.840221	-1.739598	64.806958	21.602319	0.395975	5.050826	4.647318	-4.393204	2.125778
1.8284271247	-5.695063	2.011254	-1.739059	64.788207	21.596069	0.395849	4.619006	5.080867	-4.393242	2.123973
1.732050808	-5.601660	2.100623	-1.732051	64.543247	21.514416	0.394201	4.393825	5.328809	-4.393825	2.100623
1.50	-5.151193	2.298090	-1.697468	63.296844	21.098948	0.385847	3.887553	5.955966	-4.399331	1.988006
1.40	-4.873013	2.376043	-1.675421	62.469166	20.823055	0.380325	3.681063	6.247404	-4.405235	1.918461

FOR 3θ=35°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-4.552845	2.449960	-1.649337	61.456209	20.485403	0.373594	3.479710	6.557808	-4.414781	1.838419
1.191753595	-4.165899	2.525584	-1.616715	60.137386	20.045795	0.364876	3.266191	6.921765	-4.430864	1.741682
1.10	-3.810546	2.586215	-1.585593	58.825316	19.608439	0.356250	3.087720	7.259553	-4.450788	1.652844
1.00	-3.400415	2.648755	-1.548132	57.176700	19.058900	0.345478	2.894541	7.666930	-4.481132	1.550311
0.90	-2.972297	2.707664	-1.507041	55.283250	18.427750	0.333194	2.701131	8.126394	-4.523017	1.443282
0.839099631	-2.705315	2.741781	-1.480258	54.003054	18.001018	0.324939	2.582327	8.437824	-4.555490	1.376536
0.657710346	-1.897103	2.835579	-1.392667	49.587655	16.529218	0.296768	2.216242	9.554860	-4.692776	1.174483
0.502219976	-1.209609	2.906681	-1.308279	45.074670	15.024890	0.268415	1.871059	10.829064	-4.874093	1.002610
0.466307658	-1.054086	2.921876	-1.287561	43.941838	14.647279	0.261362	1.784146	11.179428	-4.926354	0.963729
0.431357893	-0.904466	2.936218	-1.266953	42.809678	14.269893	0.254337	1.696007	11.544584	-4.981390	0.926324
0.363970234	-0.621765	2.962623	-1.225971	40.551910	13.517303	0.240398	1.514031	12.323817	-5.099751	0.855649
0.299380347	-0.359376	2.986376	-1.185134	38.310556	12.770185	0.226647	1.320909	13.176318	-5.228980	0.790052
0.267949192	-0.235220	2.997379	-1.164705	37.198955	12.399652	0.219858	1.218738	13.633256	-5.297538	0.759013
0.237004353	-0.115487	3.007852	-1.144234	36.095015	12.031672	0.213134	1.111995	14.112468	-5.368602	0.729079
0.20648339	0.000000	3.017830	-1.123691	**35**	**11 2/3**	**0.206483**	1.000000	14.615365	-5.442040	**0.700208**
0.17632698	0.111398	3.027344	-1.103048	33.915231	11.305077	0.199912	0.882024	15.143391	-5.517671	0.672358
0.00	0.700208	**3.075936**	**-0.975313**	27.706464	9.235488	0.162600	0.000000	18.917152	-5.998222	0.525156
-0.10	0.978201	3.097966	-0.897343	24.496724	8.165575	0.143489	-0.696918	21.590274	-6.253743	0.455657
-0.17632698	1.158395	3.111958	-0.835009	22.323479	7.441160	0.130608	-1.350049	23.826726	-6.393249	0.410609
-0.267949192	1.333999	3.125388	-0.756816	20.138376	6.712792	0.117699	-2.276556	26.553992	-6.430077	0.366708
-0.363970234	1.465623	3.135325	-0.670732	18.459102	6.153034	0.107805	-3.376176	29.083171	-6.221686	0.333802
-0.40	1.500108	3.137910	-0.637288	18.013518	6.004506	0.105184	-3.802869	29.832651	-6.058802	0.325181
-0.502218876	1.550366	3.141665	-0.538824	17.360087	5.786696	0.101342	-4.955698	31.000716	-5.316901	0.312616
-0.657710346	1.480131	3.136413	-0.378080	18.271922	6.090641	0.106704	-6.163880	29.393598	-3.543264	0.330175
-0.700207538	1.427609	3.132466	-0.331636	18.947611	6.315870	0.110681	-6.326334	28.301646	-2.996309	0.343305
-0.744472416	1.356762	3.127114	-0.282019	19.850450	6.616817	0.116001	-6.417788	26.957554	-2.431168	0.361017
-0.839099631	1.147683	3.111133	-0.171410	22.454665	7.484888	0.131384	-6.386611	23.679658	-1.304649	0.413287
-0.943451341	0.821030	3.085578	-0.041504	26.333111	8.777704	0.154410	-6.110051	19.983055	-0.268792	0.494950
-1.00	0.599585	3.067823	0.032799	28.824486	9.608162	0.169284	-5.907237	18.122360	0.193754	0.550311
-1.059938076	0.329231	3.045643	0.114918	31.711914	10.570638	0.186615	-5.679825	16.320501	0.615803	0.617900
-1.12369091	0.000000	3.017830	0.206483	**35**	**11 2/3**	**0.206483**	-5.442040	14.615365	1.000000	**0.700208**

FOR 3θ=35°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	-0.400617	2.982689	0.309687	38.672443	12.890814	0.228862	-5.207306	13.032704	1.353163	0.800362
-1.20	-0.452689	2.978009	0.322614	39.124210	13.041403	0.231629	-5.180688	12.856779	1.392802	0.813380
-1.30	-1.146845	2.912850	0.487773	44.622787	14.874262	0.265598	-4.894606	10.967117	1.836505	0.986919
-1.40	-1.961013	2.828605	0.672017	49.969374	16.656458	0.299186	-4.679360	9.454330	2.246151	1.190461
-1.60	-3.973386	2.559294	1.141329	59.439246	19.813082	0.360280	-4.440989	7.103622	3.167894	1.693554
-1.70	-5.183592	2.287770	1.512852	63.390251	21.130084	0.386471	-4.398775	5.919640	3.914527	1.996105
-1.732050808	-5.601660	2.100623	1.732051	64.543247	21.514416	0.394201	-4.393825	5.328809	4.393825	2.100623
-1.741341639	-5.725643	1.920982	1.920982	64.867452	21.622484	0.396382	-4.393089	4.846290	4.846290	2.131618
-1.771714948	-6.139814	IMAG.	IMAG.	65.896494	21.965498	0.403326	-4.392762	IMAG.	IMAG.	2.235161
-1.828427125	-6.949897	IMAG.	IMAG.	67.695295	22.565098	0.415545	-4.400067	IMAG.	IMAG.	2.437682
-1.9196940611	-8.356483	IMAG.	IMAG.	70.276773	23.425591	0.433269	-4.430721	IMAG.	IMAG.	2.789328
-2.00	-9.702283	IMAG.	IMAG.	72.259469	24.086490	0.447039	-4.473886	IMAG.	IMAG.	3.125778
-2.10	-11.524538	IMAG.	IMAG.	74.398941	24.799647	0.462057	-4.544890	IMAG.	IMAG.	3.581342
-2.20	-13.514806	IMAG.	IMAG.	76.224857	25.408286	0.475012	-4.631461	IMAG.	IMAG.	4.078909
-2.267949192	-14.966103	IMAG.	IMAG.	77.312127	25.770709	0.482788	-4.697605	IMAG.	IMAG.	4.441733
-2.747477419	-27.653884	IMAG.	IMAG.	82.517457	27.505819	0.520696	-5.276547	IMAG.	IMAG.	7.613679
-2.886751345	-32.200988	IMAG.	IMAG.	83.480534	27.826845	0.527839	-5.468999	IMAG.	IMAG.	8.750455
-3.100131380471	-39.982881	IMAG.	IMAG.	84.658742	28.219581	0.536635	-5.776979	IMAG.	IMAG.	10.695928
-3.732050808	-69.342301	IMAG.	IMAG.	86.826466	28.942155	0.552990	-6.748858	IMAG.	IMAG.	18.035783
-3.8284271247	-74.715729	IMAG.	IMAG.	87.046050	29.015350	0.554659	-6.902304	IMAG.	IMAG.	19.379140
-3.9953558031	-84.623024	IMAG.	IMAG.	87.380310	29.126770	0.557205	-7.170352	IMAG.	IMAG.	21.855963
-5	-161.815358	IMAG.	IMAG.	88.608047	29.536016	0.566603	-8.824523	IMAG.	IMAG.	41.154047
-5.67128182	-232.257109	IMAG.	IMAG.	89.025087	29.675029	0.569812	-9.952892	IMAG.	IMAG.	58.764485
-5.732050808	-239.457142	IMAG.	IMAG.	89.054057	29.684686	0.570036	-10.055599	IMAG.	IMAG.	60.564493
-5.846259147	-253.375468	IMAG.	IMAG.	89.105442	29.701814	0.570432	-10.248829	IMAG.	IMAG.	64.044075
-10	-1179.362054	IMAG.	IMAG.	89.806133	29.935378	0.575847	-17.365711	IMAG.	IMAG.	295.540721
-15	-3801.939881	IMAG.	IMAG.	89.939764	29.979921	0.576883	-26.001801	IMAG.	IMAG.	951.185178
-17.16933693	-5628.308593	IMAG.	IMAG.	89.959301	29.986434	0.577035	-29.754432	IMAG.	IMAG.	1407.777356
-17.18882278704	-5646.908544	IMAG.	IMAG.	89.959435	29.986478	0.577036	-29.788147	IMAG.	IMAG.	1412.427344

FOR 3θ= 30°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
17.18882279	4515.801166	IMAG.	IMAG.	-89.949223	-29.983074	-0.576956	-29.792236	IMAG.	IMAG.	-1128.372941
17.16933693	4499.767183	IMAG.	IMAG.	-89.949042	-29.983014	-0.576955	-29.758535	IMAG.	IMAG.	-1124.364445
14	2363.095392	IMAG.	IMAG.	-89.902921	-29.967640	-0.576597	-24.280370	IMAG.	IMAG.	-590.196498
11.45945741782	1243.593885	IMAG.	IMAG.	-89.815367	-29.938456	-0.575919	-19.897691	IMAG.	IMAG.	-310.321121
11.4300523	1233.293223	IMAG.	IMAG.	-89.813822	-29.937941	-0.575907	-19.847046	IMAG.	IMAG.	-307.745956
10	797.372270	IMAG.	IMAG.	-89.711745	-29.903915	-0.575116	-17.387784	IMAG.	IMAG.	-198.765717
8.595086218	481.802313	IMAG.	IMAG.	-89.522041	-29.840680	-0.573649	-14.983189	IMAG.	IMAG.	-119.873228
8.555546781	474.372743	IMAG.	IMAG.	-89.514519	-29.838173	-0.573591	-14.915775	IMAG.	IMAG.	-118.015835
7	237.706861	IMAG.	IMAG.	-89.026493	-29.675498	-0.569823	-12.284510	IMAG.	IMAG.	-58.849365
5.8462591466	123.657055	IMAG.	IMAG.	-88.112035	-29.370678	-0.562797	-10.387865	IMAG.	IMAG.	-30.336913
5.732050809	114.806824	IMAG.	IMAG.	-87.963628	-29.321209	-0.561661	-10.205541	IMAG.	IMAG.	-28.124356
5.70	112.396020	IMAG.	IMAG.	-87.919072	-29.306357	-0.561320	-10.154640	IMAG.	IMAG.	-27.521655
5.671281820	110.262715	IMAG.	IMAG.	-87.877987	-29.292662	-0.561005	-10.109139	IMAG.	IMAG.	-26.988329
5.50	98.057813	IMAG.	IMAG.	-87.607794	-29.202598	-0.558941	-9.840044	IMAG.	IMAG.	-23.937103
5.00	67.276080	IMAG.	IMAG.	-86.476745	-28.825582	-0.550336	-9.085355	IMAG.	IMAG.	-16.241670
4.3032012473	35.279130	IMAG.	IMAG.	-83.082489	-27.694163	-0.524882	-8.198420	IMAG.	IMAG.	-8.242432
4.219331772	32.199823	IMAG.	IMAG.	-82.377841	-27.459280	-0.519664	-8.119344	IMAG.	IMAG.	-7.472605
3.8284271247	19.818351	IMAG.	IMAG.	-77.131361	-25.710454	-0.481492	-7.951171	IMAG.	IMAG.	-4.377237
3.732050808	17.237604	IMAG.	IMAG.	-75.000000	-25.000000	-0.466308	-8.003409	IMAG.	IMAG.	-3.732051
3.44991420939	10.673460	IMAG.	IMAG.	-64.441161	-21.480387	-0.393515	-8.766917	IMAG.	IMAG.	-2.091015
3.340232616	8.499384	IMAG.	IMAG.	-57.129242	-19.043081	-0.345169	-9.677097	IMAG.	IMAG.	-1.547496
3.141756715	5.066773	IMAG.	IMAG.	-34.580165	-11.526722	-0.203938	-15.405448	IMAG.	IMAG.	-0.689343
3.017830135	3.233836	IMAG.	IMAG.	-13.013088	-4.337696	-0.075852	-39.785744	IMAG.	IMAG.	-0.231109
2.886751345	1.539601	-0.577350	-0.577350	10.893395	3.631132	0.063460	45.489173	-9.097835	-9.097835	0.192450
2.747477419	0.000000	0.176327	-1.191754	**30**	**10**	**0.176327**	15.581719	1.000000	-6.758770	**0.577350**
2.267949192	-3.470054	1.094551	-1.630449	55.312628	18.437543	0.333384	6.802821	3.283157	-4.890610	1.444864
2.00	-4.350853	1.441478	-1.709427	59.011912	19.670637	0.357474	5.594816	4.032402	-4.781965	1.665064
1.8284271247	-4.585731	1.632914	-1.729290	59.881147	19.960382	0.363187	5.034391	4.496065	-4.761428	1.723783
1.732050808	-4.618802	1.732051	-1.732051	60.000000	20.000000	0.363970	4.758770	4.758770	-4.758770	1.732051
1.50	-4.444764	1.949475	-1.717424	59.364831	19.788277	0.359791	4.169087	5.418354	-4.773394	1.688541
1.40	-4.273469	2.034831	-1.702781	58.715558	19.571853	0.355531	3.937777	5.723366	-4.789407	1.645718

FOR 3θ= 30°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
1.30	-4.052816	2.115587	-1.683536	57.841945	19.280648	0.349816	3.716241	6.047715	-4.812634	1.590554
1.191753595	-3.765282	2.198070	-1.657773	56.636273	18.878758	0.341962	3.485043	6.427813	-4.847823	1.518671
1.10	-3.487431	2.264134	-1.632083	55.393081	18.464360	0.333904	3.294362	6.780798	-4.887885	1.449208
1.00	-3.154701	2.332247	-1.600196	53.793977	17.931326	0.323595	3.090282	7.207299	-4.945056	1.366025
0.90	-2.796611	2.396410	-1.564359	51.925197	17.308399	0.311626	2.888076	7.690016	-5.019987	1.276503
0.839099631	-2.568665	2.433588	-1.540637	50.648347	16.882782	0.303495	2.764789	8.018543	-5.076316	1.219517
0.657710346	-1.860522	2.535959	-1.461619	46.191502	15.397167	0.275393	2.388264	9.208520	-5.307398	1.042481
0.502219976	-1.239504	2.613857	-1.384026	41.580278	13.860093	0.246736	2.035455	10.593742	-5.609341	0.887226
0.466307658	-1.096799	2.630558	-1.364815	40.416060	13.472020	0.239562	1.946498	10.980684	-5.697119	0.851550
0.431357893	-0.958743	2.646346	-1.345653	39.250059	13.083353	0.232401	1.856093	11.386981	-5.790220	0.817036
0.363970234	-0.695796	2.675485	-1.307404	36.917511	12.305837	0.218142	1.668502	12.264879	-5.993364	0.751299
0.299380347	-0.449199	2.701795	-1.269124	34.592089	11.530696	0.204010	1.467477	13.243424	-6.220884	0.689650
0.267949192	-0.331615	2.714020	-1.249919	33.434949	11.144983	0.197008	1.360095	13.776218	-6.344519	0.660254
0.237004353	-0.217641	2.725685	-1.230638	32.283082	10.761027	0.190055	1.247028	14.341533	-6.475158	0.631761
0.20648339	-0.107143	2.736826	-1.211259	31.137687	10.379229	0.183160	1.127341	14.942299	-6.613131	0.604136
0.17632698	0.000000	2.747477	-1.191754	**30**	**10**	**0.176327**	1.000000	15.581719	-6.758770	**0.577350**
0.00	0.577350	**2.802517**	**-1.070466**	23.413224	7.804408	0.137061	0.000000	20.447174	-7.810125	0.433013
-0.10	0.859030	2.828066	-0.996015	19.930281	6.643427	0.116472	-0.858575	24.281054	-8.551530	0.362593
-0.17632698	1.046997	2.844686	-0.936308	17.515744	5.838581	0.102257	-1.724357	27.819075	-9.156449	0.315601
-0.267949192	1.237604	2.861210	-0.861210	15.000000	5.000000	0.087489	-3.062673	32.703777	-9.843672	0.267949
-0.363970234	1.391592	2.874328	-0.778307	12.922955	4.307652	0.075325	-4.832018	38.159183	-10.332699	0.229452
-0.40	1.436222	2.878093	-0.746042	12.314191	4.104730	0.071764	-5.573835	40.105041	-10.395793	0.218295
-0.502218876	1.520471	2.885156	-0.650886	11.157387	3.719129	0.065002	-7.726163	44.385397	-10.013267	0.197233
-0.657710346	1.516711	2.884842	-0.495081	11.209211	3.736404	0.065305	-10.071341	44.174802	-7.581036	0.198172
-0.700207538	1.485459	2.882228	-0.449969	11.639298	3.879766	0.067818	-10.324746	42.499215	-6.634919	0.205985
-0.744472416	1.438181	2.878258	-0.401735	12.287402	4.095801	0.071607	-10.396611	40.195082	-5.610257	0.217805
-0.839099631	1.284333	2.865212	-0.294062	14.373700	4.791233	0.083818	-10.010940	34.183623	-3.508323	0.256267
-0.943451341	1.026239	2.842866	-0.167364	17.785753	5.928584	0.103844	-9.085259	27.376267	-1.611685	0.320791
-1.00	0.845299	2.826838	-0.094788	20.103909	6.701303	0.117496	-8.510924	24.059006	-0.806730	0.366025
-1.059938076	0.620452	2.806478	-0.014490	22.891280	7.630427	0.133969	-7.911815	20.948712	-0.108156	0.422237
-1.12369091	0.342531	2.780588	0.075154	26.184149	8.728050	0.153523	-7.319386	18.111919	0.489528	0.491717

FOR 3θ= 30°		OTHER TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$z^3 - 3\zeta z^2 - 3z + \zeta = y$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = \zeta - y/4$
-1.191753595	0.000000	2.747477	0.176327	**30**	**10**	**0.176327**	-6.758770	15.581719	1.000000	**0.577350**
-1.20	-0.044803	2.743041	0.189010	30.478981	10.159660	0.179202	-6.696367	15.307010	1.054731	0.588551
-1.30	-0.646816	2.680786	0.351265	36.466408	12.155469	0.215394	-6.035444	12.445949	1.630800	0.739054
-1.40	-1.361469	2.599256	0.532795	42.543150	14.181050	0.252687	-5.540451	10.286463	2.108519	0.917718
-1.60	-3.152700	2.332628	0.999423	53.783975	17.927992	0.323531	-4.945433	7.209911	3.089111	1.365525
-1.70	-4.241277	2.048075	1.383976	58.590766	19.530255	0.354713	-4.792608	5.773893	3.901679	1.637669
-1.732050808	-4.618802	1.732051	1.732051	60.000000	20.000000	0.363970	-4.758770	4.758770	4.758770	1.732051
-1.741341639	-4.730891	IMAG.	IMAG.	60.396570	20.132190	0.366585	-4.750169	IMAG.	IMAG.	1.760073
-1.771714948	-5.105734	IMAG.	IMAG.	61.655924	20.551975	0.374919	-4.725594	IMAG.	IMAG.	1.853784
-1.828427125	-5.840565	IMAG.	IMAG.	63.858215	21.286072	0.389604	-4.693044	IMAG.	IMAG.	2.037492
-1.9196940611	-7.121070	IMAG.	IMAG.	67.015433	22.338478	0.410915	-4.671759	IMAG.	IMAG.	2.357618
-2.00	-8.350853	IMAG.	IMAG.	69.432624	23.144208	0.427448	-4.678927	IMAG.	IMAG.	2.665064
-2.10	-10.021994	IMAG.	IMAG.	72.028217	24.009406	0.445425	-4.714594	IMAG.	IMAG.	3.082849
-2.20	-11.853776	IMAG.	IMAG.	74.229126	24.743042	0.460859	-4.773694	IMAG.	IMAG.	3.540794
-2.267949192	-13.193176	IMAG.	IMAG.	75.532011	25.177337	0.470081	-4.824590	IMAG.	IMAG.	3.875644
-2.747477419	-24.994528	IMAG.	IMAG.	81.665510	27.221837	0.514412	-5.341005	IMAG.	IMAG.	6.825982
-2.886751345	-29.252414	IMAG.	IMAG.	82.777102	27.592367	0.522618	-5.523638	IMAG.	IMAG.	7.890454
-3.100131380471	-36.563463	IMAG.	IMAG.	84.124968	28.041656	0.532642	-5.820287	IMAG.	IMAG.	9.718216
-3.732050808	-64.331615	IMAG.	IMAG.	86.565051	28.855017	0.551006	-6.773161	IMAG.	IMAG.	16.660254
-3.8284271247	-69.436483	IMAG.	IMAG.	86.808931	28.936310	0.552857	-6.924808	IMAG.	IMAG.	17.936471
-3.9953558031	-78.862418	IMAG.	IMAG.	87.178850	29.059617	0.555670	-7.190159	IMAG.	IMAG.	20.292955
-5	-152.723920	IMAG.	IMAG.	88.522045	29.507348	0.565942	-8.834826	IMAG.	IMAG.	38.758330
-5.67128182	-220.525430	IMAG.	IMAG.	88.971622	29.657207	0.569400	-9.960093	IMAG.	IMAG.	55.708708
-5.732050808	-227.470054	IMAG.	IMAG.	89.002696	29.667565	0.569640	-10.062586	IMAG.	IMAG.	57.444864
-5.846259147	-240.901003	IMAG.	IMAG.	89.057760	29.685920	0.570064	-10.255438	IMAG.	IMAG.	60.802601
-10	-1142.627730	IMAG.	IMAG.	89.799830	29.933277	0.575799	-17.367184	IMAG.	IMAG.	286.234283
-15	-3719.134081	IMAG.	IMAG.	89.938416	29.979472	0.576873	-26.002272	IMAG.	IMAG.	930.360871
-17.16933693	-5519.781593	IMAG.	IMAG.	89.958497	29.986166	0.577028	-29.754753	IMAG.	IMAG.	1380.522749
-17.18882278704	-5538.134787	IMAG.	IMAG.	89.958635	29.986212	0.577029	-29.788468	IMAG.	IMAG.	1385.111047

15.3. Determination of Whether A *Given Cubic Equation* Contains 3 *Real Roots*.

A determination of whether any *given Cubic Equation* contains *three real roots* is accomplished by conducting a comparison between its coefficients and those expressed in the *Equation 42* cited below:

$$u^3 + (3V)u^2 + 3(V^2 - f^2)u + V^3 - 3f^2V - 2\psi f^3 = 0$$

Letting $u = R$ renders,

$$R^3 + (3V)R^2 + 3(V^2 - f^2)R + V^3 - 3f^2V - 2\psi f^3 = 0$$

For this analysis, the *given Cubic Equation* shall appear in the form of the *Characteristic Cubic Equation* reiterated as follows:

$$AR^3 + BR^2 + CR + D = 0 \qquad \text{(ref. } \textit{Equation 31}\text{)}$$

Such comparison of terms reveals that:

$$V = \frac{B}{3}$$

$$f = \sqrt{V^2 - \frac{C}{3}}$$

$$\psi = \frac{V^3 - 3f^2V - D}{2f^3} = \cos(6\omega)$$

Quite simply, if the cos (6ω) lies within the range $-1 \le \psi \le +1$, then the given Cubic Equation contains three real roots.

As an example, consider whether or not the following *given Cubic Equation* contains three real roots:

$$R^3 + 6R^2 + 11.19295467R + 5.866850998 = 0$$

Where,

$$V = \frac{B}{3} = \frac{6}{3} = 2$$

$$f = \sqrt{V^2 - \frac{C}{3}} = \sqrt{2^2 - \frac{11.19295467}{3}} = \sqrt{0.26901511} = 0.518666666$$

$$\psi = \frac{V^3 - 3f^2V - D}{2f^3}$$

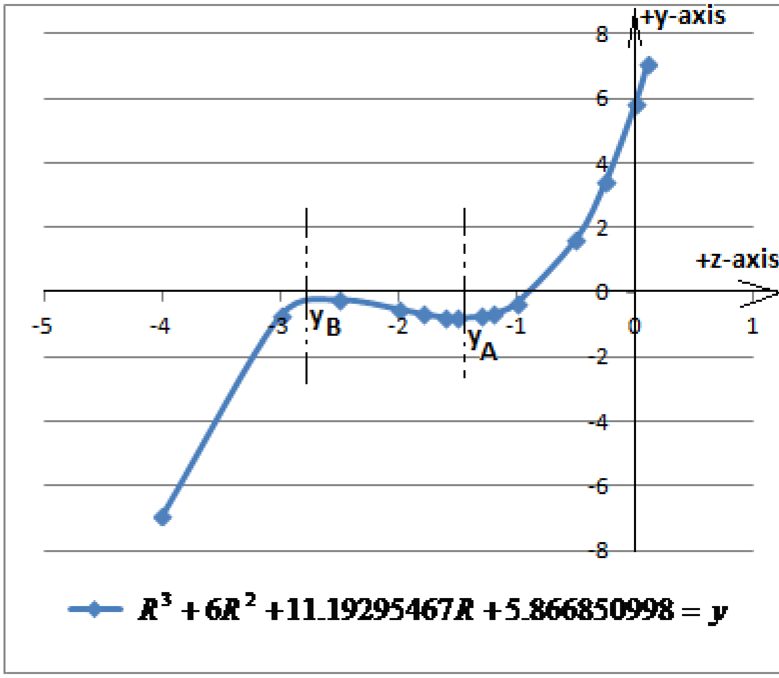
$$= \frac{2^3 - 3(0.518666666)^2(2) - 5.866850998}{2(0.518666666)^3}$$

$$= +1.8600352$$

$$= \cos(6\omega)$$

Since cos (6ω)is greater than unity, the *given Cubic Equation* does not contain *three real roots*. This is illustrated in *Figure 38* which shows that the z-axis lies above y_B. Hence the *Cubic Curve* illustrated in this figure intercepts the z-axis only at *one* location.

Figure 38. Graph and Plot of Given Cubic Function.



z	y
0.1	7.047146465
0	5.866850998
-0.25	3.427987331
-0.5	1.645373663
-1	-0.326103672
-1.2	-0.652694606
-1.3	-0.740990073
-1.5	-0.797581007
-1.6	-0.777876474
-1.8	-0.672467408
-2	-0.519058342
-2.5	-0.240535677
-3	-0.712013012
-4	-6.904967682

The three curves illustrated in *Figure 39* include the one shown in *Figure 38* *along with* two additional *Cubic Functions* which are of the same shape, but displaced upwards. Their associated equations remain the same with the only exception being that the 'D' coefficients are increased from 5.866850998 to 6.5 and 8, respectively, thereby elevating the resulting curves accordingly.

As indicated, the z-axis intercepts the middle curve in three locations, thereby rendering three real roots. As shown, the upper and lower curves possess just one real root each.

RST values are computed for the middle curve by realizing that 'V' and 'f' remain the same.

Then,

$$\psi = \frac{V^3 - 3f^2V - D}{2f^3}$$

$$= \frac{2^3 - 3(0.518666666)^2(2) - 6.5}{2(0.518666666)^3}$$

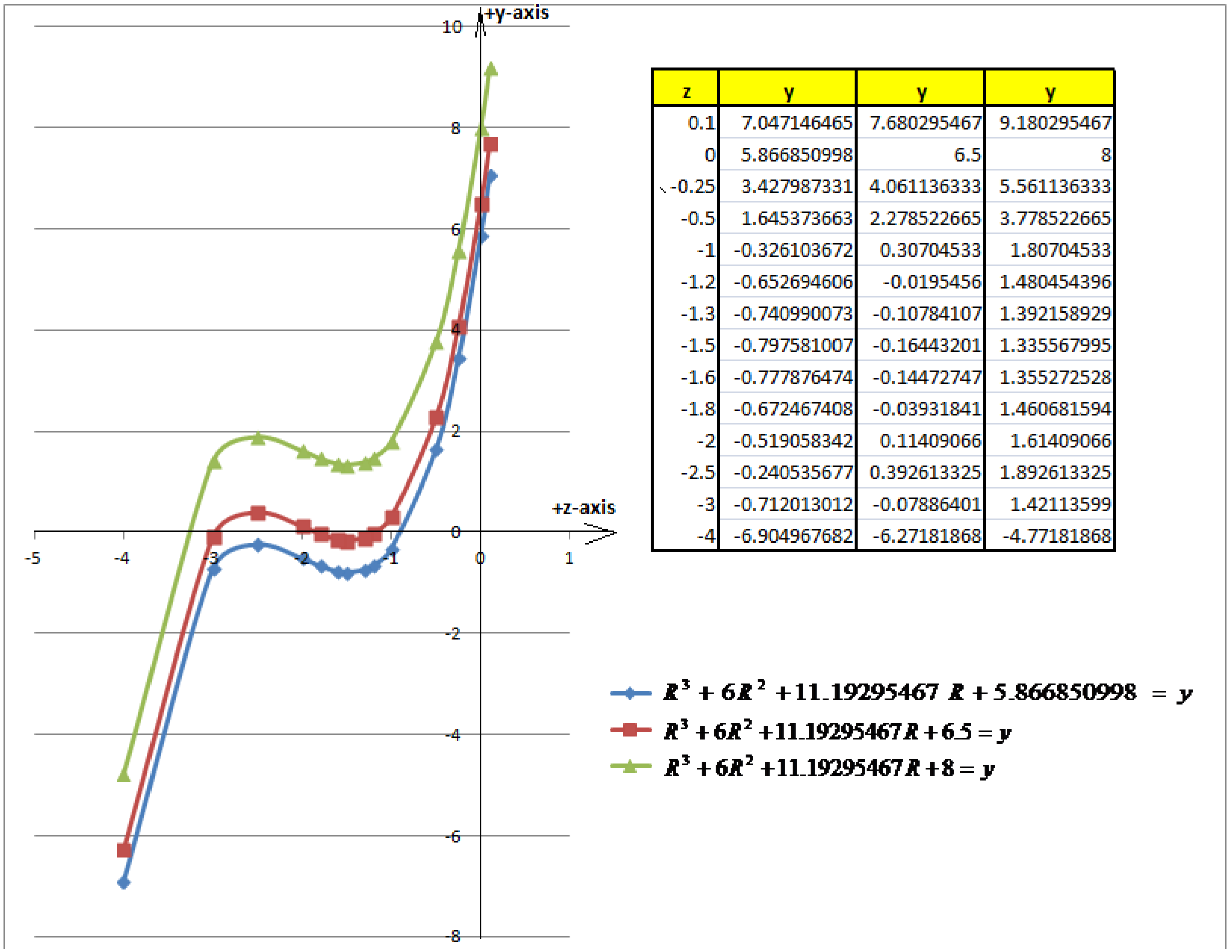
$$= -0.40884161$$

$$= \cos(6\varpi)$$

$$6\varpi = +114.1320876^o; -114.1320876^o; +114.1320876^o + 360^o$$

$$= +114.1320876^o; 245.8679124^o; +474.1320876^o$$

Figure 39. Graph and Plot of Three Given Cubic Functions.



z	y	y	y
0.1	7.047146465	7.680295467	9.180295467
0	5.866850998	6.5	8
-0.25	3.427987331	4.061136333	5.561136333
-0.5	1.645373663	2.278522665	3.778522665
-1	-0.326103672	0.30704533	1.80704533
-1.2	-0.652694606	-0.0195456	1.480454396
-1.3	-0.740990073	-0.10784107	1.392158929
-1.5	-0.797581007	-0.16443201	1.335567995
-1.6	-0.777876474	-0.14472747	1.355272528
-1.8	-0.672467408	-0.03931841	1.460681594
-2	-0.519058342	0.11409066	1.61409066
-2.5	-0.240535677	0.392613325	1.892613325
-3	-0.712013012	-0.07886401	1.42113599
-4	-6.904967682	-6.27181868	-4.77181868

$$2\varpi = \frac{+114.1320876^o}{3}; \frac{245.8679124^o}{3}; \frac{+474.1320876^o}{3}$$

$$= 38.04402919^o; 81.9559708^o; 158.0440292^o$$

$$\cos 2\varpi = 0.787537413; 0.139934035; -0.927471448$$

$$
\begin{aligned}
2f\cos(2\omega) &= (2f)[0.787537413;0.139934035;-0.927471448] \\
&= (1.037333332)[0.787537413;0.139934035;-0.927471448] \\
&= +0.816938808;0.145158238-;0.962097047 \\
&= \ell
\end{aligned}
$$

Where,

$$
\begin{aligned}
u &= \ell - V \\
&= [+0.816938808;0.145158238;-0.962097047] - (2) \\
&= -1.183061192;-1.854841762;-2.9620970471 \\
&= R,S,T
\end{aligned}
$$

Check:

$$R^3 + 6R^2 + 11.19295467R + 6.5 = (-1.183061192)^3 + 6(-1.183061192)^2 + 11.19295467(-1.183061192) + 6.5$$

$$
\begin{aligned}
&= -1.655852413 + 8.397802704 - 13.24195029 + 6.5 \\
&= 0
\end{aligned}
$$

$$S^3 + 6S^2 + 11.19295467S + 6.5 = (-1.854841762)^3 + 6(-1.854841762)^2 + 11.19295467(-1.854841762) + 6.5$$

$$
\begin{aligned}
&= -6.381468011 + 20.64262777 - 20.76115976 + 6.5 \\
&= 0
\end{aligned}
$$

$$T^3 + 6T^2 + 11.19295467T + 6.5 = (-2.9620970471)^3 + 6(-2.9620970471)^2 + 11.19295467(-2.9620970471) + 6.5$$

$$
\begin{aligned}
&= -25.98949552 + 52.64411349 - 33.15461798 + 6.5 \\
&= 0
\end{aligned}
$$

The above postulated theory implies that any *given Cubic Equation*, or transformed *Cubic Function*, is comprised of *three real roots* when it possesses real S and T values. If not, it is manifest by only *one* real root, accompanied by two other imaginary roots.

Figure 19 discloses a *Cubic Curve* which possesses *three real roots* that are identified by small triangles appearing on the z-axis. Were this curve to be elevated by a value of y_A, it would possess three different roots, one of which would equal z_A. Relativistically, or relatively speaking, this is *equivalent* to lowering the z-axis by a distance of y_A such that it intersects point A.

Such elevation, or relativistic lowering, characterizes the *limit* of the three root representation because placing an abscissa either below point A or above point B would establish a new condition such that the z-axis then would cross the resident cubic curve only at *one* location. Hence, the *three real root regime* consists only of conditions where the z-axis resides either on or between points A and B. Otherwise, the two remaining roots are *said* to be *imaginary*. Quite frankly, they don't exist! In other words, the *imaginary nomenclature* merely represents a *convenience for mathematical closure*, but one of practically *useless value* since it merely modifies a *non-existent manifestation*.

In practice, the above methodology is best demonstrated by considering the *3θ Cosine Cubic Equation*, or one of its associated *Transformed Cubic Functions*. As such, hereinafter, the *3θ Cubic Equation* just signifies a short-hand form for the *3θ Tangent Cubic Equation.*

The *3θ Cosine Cubic Equation* is given below, where the cos θ is replaced by z (*ref. Equation 1)*:

$$z^3 - \frac{3}{4}z - \frac{\tau}{4} = 0$$

Its *associated Transformed Cubic Function* is developed as follows:

$$z^3 - \frac{3}{4}z - \frac{\tau}{4} = y$$

$$z^3 - \frac{3}{4}z - (\frac{\tau}{4} + y) = 0$$

Just as before, respective z_1 and z_2 values are determined as:

$$(z - z_f)(z^2 + Mz + N) = 0$$

$$z^3 + (M - z_f)z^2 + (N - Mz_f)z - Nz_f = 0$$

Where,

$$z^3 - 0z^2 - (3/4z) - (\tau/4 + y) = 0$$

Comparing respective second and third term coefficients from the last two equations yields:

$$M - z_f = 0$$

$$M = +z_f$$

$$N - Mz_f = -3/4$$

$$N = -3/4 + z_f^{\ 2}$$

Solving the remaining *Quadratic Equation* via *Quadratic Formula* gives the following expression for the other two roots:

$$z_1, z_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-M \pm \sqrt{M^2 - 4N}}{2}$$

$$= \frac{-z_f \pm \sqrt{z_f^{\ 2} - 4(1)[-3/4 + z_f^{\ 2}]}}{2(1)}$$

$$= \frac{1}{2}[-z_f \pm \sqrt{3(1 - z_f^{\ 2})}]$$

Furthermore,

$$\zeta' = \frac{\delta' - \beta'}{1 - \gamma'}.$$

$$= \frac{-(\tau/4 + y) - 0}{1 - (-3/4)}$$

$$= \frac{-(\tau + 4y)}{7}$$

Figure 40 presents the graph for the *3θ Cosine Cubic Function* and its *associated R, S and T Curves* for the condition where $\tau = 1/2$, such that when $y = 0$:

$$\zeta = \frac{\delta - \beta}{1 - \gamma}.$$

$$= \frac{-(\tau/4 + y) - 0}{1 - (-3/4)}$$

$$= \frac{-(\tau/4 + 0) - 0}{1 - (-3/4)}$$

$$= \frac{-\tau}{7}$$

$$= \frac{-1}{14}$$

Then, 3θ equals arc tan -1/14, or -4.085617° which is equivalent to:

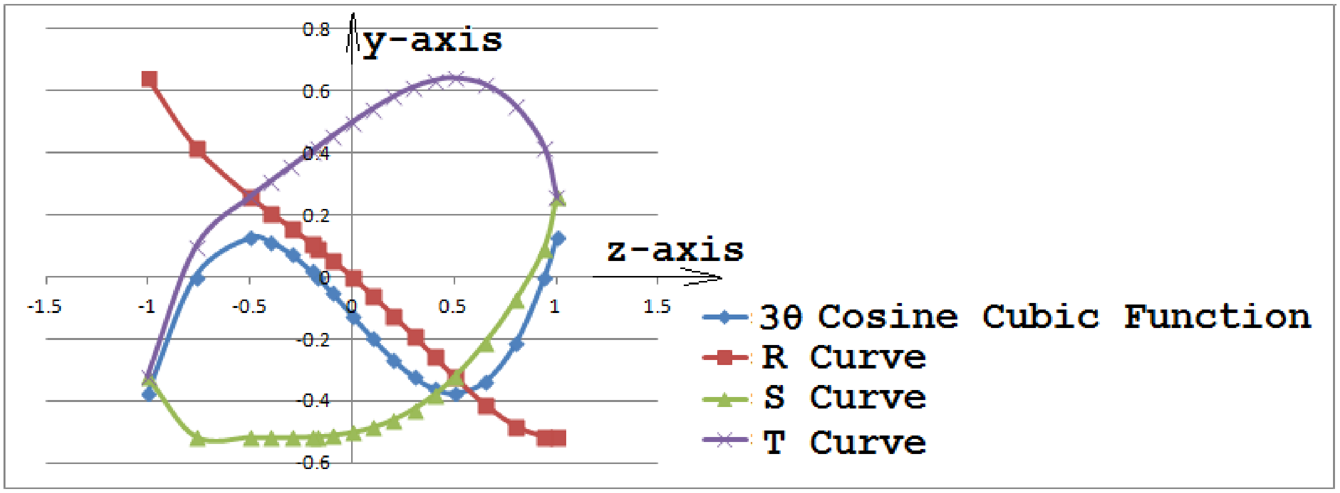
360° -4.085617° = 355.9143832°

Table 32 gives the associated mapping where S and T are shown to be imaginary when:

$z = z_f < -1$

$z = z_f > +1$

Figure 40. 3θ Cosine Cubic Function and its Associated R, S and T Curves.



In *Figure 40* notice that the *S and T Curves* exist between minus one and plus one, which turns out to be the same exact regime for which the cosine θ applies. This is verified by examining the z_1 and z_2 roots via the equation below, and realizing that the square root radical is real only when z_f exhibits a value between −1 thru +1.

$$z_1, z_2 = \frac{1}{2}[-z_f \pm \sqrt{3(1-z_f{}^2)}]$$

Just as before, the above illustrated *S and T Curves* join at their *real root* boundaries.

Table 32. 3θ Cosine Cubic Function Plot.

		REMAINING TWO ROOTS		θ' FUNCTIONS			RST FUNCTIONS			tan (3θ')
$z = z_f$	$y = z^3 - 3/4z - \tau/4$	z_1	z_2	$3\theta'$	θ'	$\tan\theta'$	R	S	T	$\zeta' = -(\tau+4y)/7$
4	60.875	IMAG.	IMAG.	271.643281	90.547760	-104.596924	-0.038242	IMAG.	IMAG.	-34.857143
3	24.625	IMAG.	IMAG.	274.044486	91.348162	-42.491341	-0.070603	IMAG.	IMAG.	-14.142857
2.5	13.625	IMAG.	IMAG.	277.253195	92.417732	-23.684089	-0.105556	IMAG.	IMAG.	-7.857143
2	6.375	IMAG.	IMAG.	285.068488	95.022829	-11.377836	-0.175780	IMAG.	IMAG.	-3.714286
1.5	2.125	IMAG.	IMAG.	307.874984	102.624995	-4.464593	-0.335977	IMAG.	IMAG.	-1.285714
1.25	0.890625	IMAG.	IMAG.	329.870957	109.956986	-2.753908	-0.453900	IMAG.	IMAG.	-0.580357
1	**0.125**	-0.500000	-0.500000	351.869898	117.289966	-1.938297	-0.515917	0.257958	0.257958	-0.142857
0.93969	0	-0.173648	-0.766044	**355.914383**	**118.638128**	**-1.831229**	-0.513149	0.094826	0.418323	**-0.071429**
0.8	-0.213	0.119615	-0.919615	362.878734	120.959578	-1.666942	-0.479921	-0.071757	0.551678	0.050286
0.65	-0.337875	0.333122	-0.983122	366.935548	122.311849	-1.581120	-0.411101	-0.210688	0.621789	0.121643
0.5	-0.375	0.500000	-1.000000	368.130102	122.710034	-1.557060	-0.321118	-0.321118	0.642236	0.142857
0.4	-0.361	0.593725	-0.993725	367.680409	122.560136	-1.566056	-0.255419	-0.379121	0.634540	0.134857
0.3	-0.323	0.676136	-0.976136	366.455157	122.151719	-1.590945	-0.188567	-0.424990	0.613557	0.113143
0.2	-0.267	0.748528	-0.948528	364.638980	121.546327	-1.628894	-0.122783	-0.459532	0.582314	0.081143
0.1	-0.199	0.811684	-0.911684	362.421350	120.807117	-1.677042	-0.059629	-0.483998	0.543626	0.042286
0	-0.125	0.866025	-0.866025	360.000000	120.000000	-1.732051	0.000000	-0.500000	0.500000	0.000000
-0.1	-0.051	0.911684	-0.811684	357.578650	119.192883	-1.789811	0.055872	-0.509375	0.453503	-0.042286
-0.1736	0	0.939693	-0.766044	**355.914383**	**118.638128**	**-1.831229**	0.094826	-0.513149	0.418323	**-0.071429**
-0.2	0.017	0.948528	-0.748528	355.361020	118.453673	-1.845327	0.108382	-0.514016	0.405634	-0.081143
-0.3	0.073	0.976136	-0.676136	353.544843	117.848281	-1.892801	0.158495	-0.515710	0.357214	-0.113143
-0.4	0.111	0.993725	-0.593725	352.319591	117.439864	-1.925915	0.207694	-0.515976	0.308282	-0.134857
-0.5	0.125	1.000000	-0.500000	351.869898	117.289966	-1.938297	0.257958	-0.515917	0.257958	-0.142857
-0.766	0	0.939693	-0.173648	**355.914383**	**118.638128**	**-1.831229**	0.418323	-0.513149	0.094826	**-0.071429**
-1	-0.375	0.500000	0.500000	368.130102	122.710034	-1.557060	0.642236	-0.321118	-0.321118	0.142857
-1.25	-1.140625	IMAG.	IMAG.	390.129043	130.043014	-1.189938	1.050475	IMAG.	IMAG.	0.580357
-1.5	-2.375	IMAG.	IMAG.	412.125016	137.375005	-0.920353	1.629810	IMAG.	IMAG.	1.285714
-2	-6.625	IMAG.	IMAG.	434.931512	144.977171	-0.700802	2.853875	IMAG.	IMAG.	3.714286
-2.5	-13.875	IMAG.	IMAG.	442.746805	147.582268	-0.635053	3.936676	IMAG.	IMAG.	7.857143
-3	-24.875	IMAG.	IMAG.	445.955514	148.651838	-0.609161	4.924803	IMAG.	IMAG.	14.142857
-4	-61.125	IMAG.	IMAG.	448.356719	149.452240	-0.590168	6.777727	IMAG.	IMAG.	34.857143

15.4. Matching All *GCF's* to *3θ Cubic Functions* of the Same Shape.

It is now possible to identify a *3θ Cubic Function* which assumes the exact same shape as any *given Generalized Cubic Function (GCF)*.

This analysis *begins* with two curves depicted as follows, already proven to be of the *same exact curve shape (Ref. Section 14.2 and 14.2.1)*:

$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$ *[Given Generalized Cubic Curve]*

$z'^3 + \sigma z'^2 + \nu = y'$ *[Displaced family curve]*

Where,

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$\nu = \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta']$$

$$= \frac{1}{27}[2\beta'^3 + 2(\beta'^2 - 3\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta']$$

$$= \frac{1}{27}[2\beta'^3 - 2\sigma^3 - 9\beta'\gamma' + 27\delta']$$

Representing a specific case of the *Generalized Cubic Function*, the *3θ Cubic Function* is *redeveloped* below:

Where,

$$z''^3 = 3z'' - \zeta(1 - 3z''^2) \quad \textit{[Ref. Equation 22]}$$

$$z''^3 - 3\zeta z''^2 - 3z'' + \zeta = 0$$

$$z''^3 - 3\zeta z''^2 - 3z'' + \zeta = y''$$

$$z''^3 - 3\zeta z''^2 - 3z'' + \zeta = y''$$

$z''^3 + \beta_{3\theta} z''^2 + \gamma_{3\theta} z'' + \delta_{3\theta} = y''$ [*Associated 3θ Cubic Curve*]

Such that,

$$\beta_{3\theta} = -3\zeta$$
$$\gamma_{3\theta} = -3$$
$$\delta_{3\theta} = \zeta$$

Associated $\sigma_{3\theta}$ and $\nu_{3\theta}$ coefficients then are determined as:

$$\sigma_{3\theta} = -\sqrt{\beta_{3\theta}^{\ 2} - 3\gamma_{3\theta}}$$

$$= -\sqrt{(3\zeta)^2 - 3(-3)}$$

$$\begin{aligned}
\nu_{3\theta} &= \frac{1}{27}[2\beta_{3\theta}{}^3 - 2\sigma_{3\theta}{}^3 - 9\beta_{3\theta}\gamma_{3\theta} + 27\delta_{3\theta}] \\
&= \frac{1}{27}[2(-3\zeta)^3 - 2(-3\sqrt{\zeta^2+1})^3 - 9(-3\zeta)(-3) + 27\zeta] \\
&= \frac{1}{27}[-2(27)\zeta^3 + 2(27)\sqrt{\zeta^2+1}^3 + 27\zeta(-3) + 27\zeta] \\
&= -2\zeta^3 + 2\sqrt{\zeta^2+1}^3 - 3\zeta + \zeta \\
&= -2\zeta^3 - 2\zeta + 2\sqrt{\zeta^2+1}^3 \\
&= 2[\sqrt{\zeta^2+1}^3 - \zeta(\zeta^2+1)] \\
&= 2[(\zeta^2+1)\sqrt{\zeta^2+1} - \zeta(\zeta^2+1)] \\
&= 2(\zeta^2+1)(\sqrt{\zeta^2+1} - \zeta)
\end{aligned}$$

As such:

$z'''^3 + \sigma_{3\theta} z'''^2 + \nu_{3\theta} = y'''$ *[Associated 3θ Displaced family curve]*

Lastly, *Generalized Cubic family curves* exhibit the *same exact curve shapes* as their *associated parent curves*.

As this pertains to such functions which are *devoid* of their third terms, this is easily observed by comparing the following two *family functions* in the same exact manner as was previously done in *Section 14.2.2*:

$z'^3 + \sigma z'^2 + \nu = y'$ *[Displaced family curve]*

$z'^3 + \sigma z'^2 = y''$ *[Parent Curve]*

Accordingly, the *curve set* $z'^3 + \sigma z'^2 + \nu = y'$ retains the same shape as its *parent curve* $z'^3 + \sigma z'^2 = y''$, no matter what value of ν it contains.

Therefore, the value $\sigma_{3\theta}$ is <u>equated</u> to any predetermined coefficient σ, in order to determine an *associated 3θ Cubic Function* which assumes that *curve shape*, as follows:

$$\begin{aligned}
\sigma_{3\theta} &= \sigma \\
-\sqrt{(3\zeta)^2 - 3(-3)} &= -\sqrt{\beta'^2 - 3\gamma'} \\
9\zeta^2 + 9 &= \beta'^2 - 3\gamma' \\
\zeta^2 &= \frac{\beta'^2 - 3\gamma' - 9}{9} \\
\zeta &= \pm\frac{1}{3}\sqrt{\beta'^2 - 3\gamma' - 9}
\end{aligned}$$

As an illustrative example, consider the *given Generalized Cubic Function:*

$$z^3 + \beta' z^2 + \gamma' z + \delta' = y_{TRANSFORMED}$$

$$z^3 + 1.4z^2 - 3.2z - 0.84 = y_{TRANSFORMED}$$

Its *Displaced family function* is calculated as follows:

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$
$$= -\sqrt{1.4^2 - 3(-3.2)}$$
$$= -3.4$$

$$\nu = \frac{1}{27}[2\beta'^3 - 2\sigma^3 - 9\beta'\gamma' + 27\delta']$$
$$= \frac{1}{27}[2(1.4)^3 - 2(-3.4)^3 - 9(1.4)(-3.2) + 27(-0.84)]$$
$$= \frac{1}{27}(5.488 + 78.608 + 40.32 - 22.68)$$
$$= \frac{1}{27}(101.736)$$
$$= 3.768$$

Hence, the *associated displaced family function* for the *given Generalized Cubic Function* $z^3 + 1.4z^2 - 3.2z - 0.84 = y$ is established as:

$$z'^3 + \sigma z'^2 + \nu = y'$$

$$z'^3 - 3.4z'^2 + 3.768 = y'$$

Next, the *associated 3θ Cubic Function* for the *given Generalized Cubic Function* $z^3 + 1.4z^2 - 3.2z - 0.84 = y$ is developed as follows:

$$\zeta = \pm\frac{1}{3}\sqrt{\beta'^2 - 3\gamma' - 9}$$
$$= \pm\frac{1}{3}\sqrt{1.4^2 - 3(-3.2) - 9}$$
$$= \pm\frac{1.6}{3}$$

$$z''^3 + \beta_{3\theta} z''^2 + \gamma_{3\theta} z'' + \delta_{3\theta} = y''$$

$$z''^3 - 3\zeta z''^2 - 3z'' + \zeta = y$$

$$z''^3 \mp 1.6z''^2 - 3z'' \pm \frac{1.6}{3} = y$$ [*Associated 3θ Cubic Curve*]

Lastly, the *Displaced family function* for the *associated 3θ Cubic Function* is determined as follows:

$$\sigma_{3\theta} = -\sqrt{\beta_{3\theta}{}^{2} - 3\gamma_{3\theta}}$$
$$= -\sqrt{(\mp 1.6)^{2} - 3(-3)}$$
$$= -3.4$$

Accordingly it is verified that the two resulting *parent curves* are of the *same exact shape* since they both possess the same value for their second term *coefficients* as follows:

$$\sigma_{3\theta} = \sigma = -3.4$$

$$\nu_{3\theta} = 2(\zeta^{2} + 1)(\sqrt{\zeta^{2} + 1} - \zeta)$$
$$= 2(\frac{2.56}{9} + 1)(\sqrt{\frac{11.56}{9}} \mp \frac{1.6}{3})$$
$$= (\frac{23.12}{9})(\frac{3.4}{3} \mp \frac{1.6}{3})$$
$$= (\frac{23.12}{9})(\frac{1.8}{3});(\frac{23.12}{9})(\frac{5}{3})$$
$$= 1.54133333;4.281481481$$

$$z'''^{3} + \sigma_{3\theta} z'''^{2} + \nu_{3\theta} = y'''$$ *[Associated 3θ Displaced family curve]*

$$z'''^{3} - 3.4z'''^{2} + 1.54133333 = y'''$$
$$z'''^{3} - 3.4z'''^{2} + 4.281481481 = y'''$$

Figure 41 shows the *relative placements* of the *following* four identically shaped associated curves.

$$z^{3} + \beta' z^{2} + \gamma' z + \delta' = y_{TRANSFORMED}$$
$$z^{3} + 1.4z^{2} - 3.2z - 0.84 = y_{TRANSFORMED}$$ *[Given Generalized Cubic Curve]*

$$z'^{3} + \sigma z'^{2} + \nu = y'$$
$$z'^{3} - 3.4z'^{2} + 3.768 = y'$$ *[Associated Displaced Family Curve]*

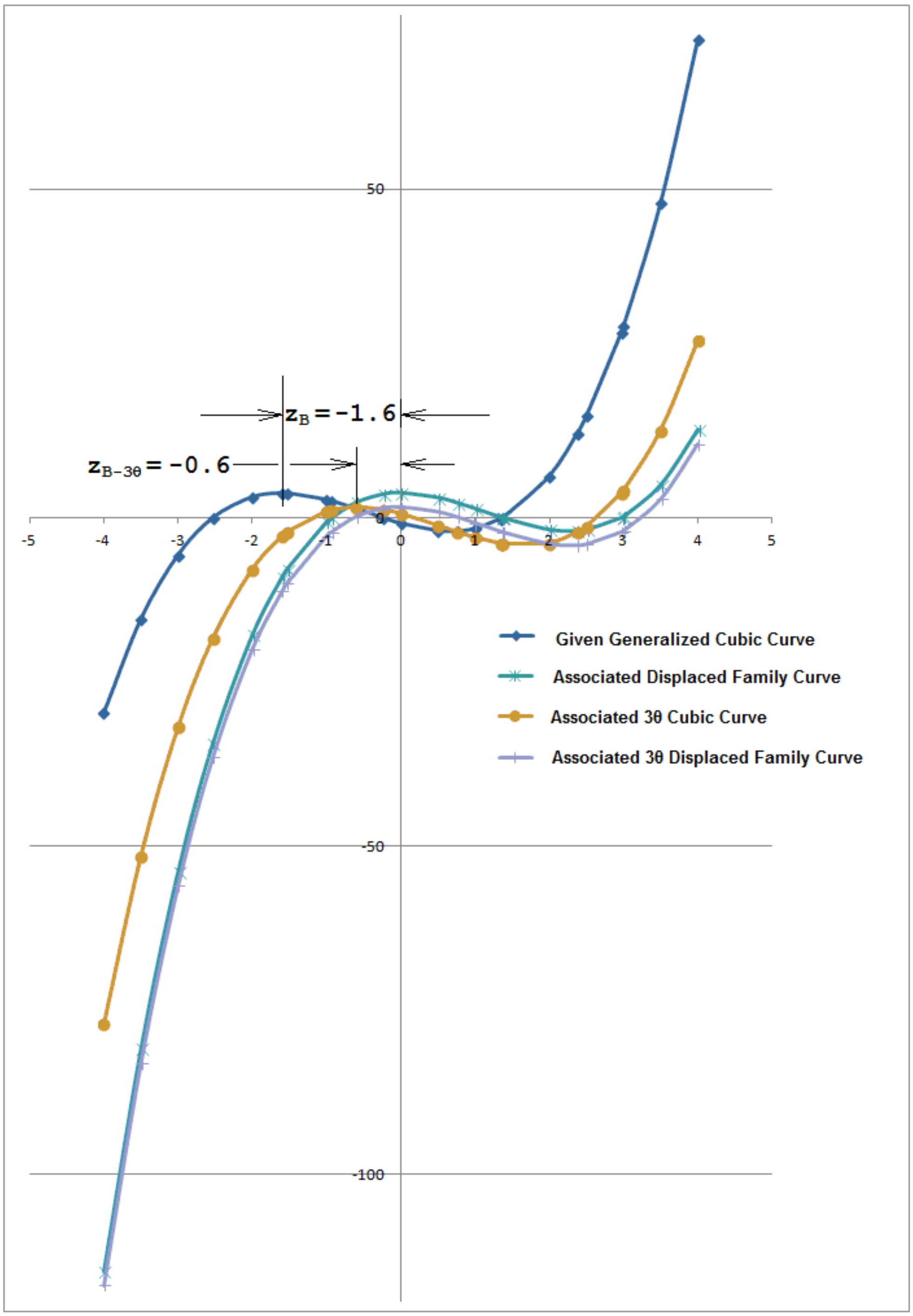
$$z''^{3} + \beta_{3\theta} z''^{2} + \gamma_{3\theta} z'' + \delta_{3\theta} = y''$$
$$z''^{3} - 3\zeta z''^{2} - 3z'' + \zeta = y''$$
$$z''^{3} - 1.6z''^{2} - 3z'' + \frac{1.6}{3} = y''$$ *[Associated 3θ Cubic Curve]*

$$z'''^{3} + \sigma_{3\theta} z'''^{2} + \nu_{3\theta} = y'''$$
$$z'''^{3} - 3.4z'''^{2} + 1.54133333 = y'''$$ *[Associated 3θ Displaced Family Curve]*

Therein, notice that:

- For the *given Generalized Cubic Curve*, $y_B = v$. At $z = 0$, $y = v$ upon the *associated Displaced Family Curve*
- For the *associated 3θ Cubic Curve*, $y_B = v_{3\theta}$. At $z = 0$, $y = v_{3\theta}$ upon the *associated 3θ Displaced Family Curve*

Figure 41. ***Given Cubic Function*** **Exhibiting Same Shape as** ***3θ Cubic Function.***



For the *given Generalized Cubic Function*

$z^3 + 1.4z^2 - 3.2z - 0.84 = y$:

$$z_B = \frac{1}{3}[-\beta' - \sqrt{\beta'^2 - 3\gamma'}] \qquad \textit{[Ref. Section 14.2.1]}$$

$$= +\frac{1}{3}(-1.4 - \sqrt{1.4^2 - 3(-3.2)}) = -1.6$$

For the *associated 3θ Cubic Function* $z''^3 - 1.6z''^2 - 3z'' + \frac{1.6}{3} = y''$:

$$z_{B-3\theta} = +\frac{1}{3}(-\beta_{3\theta} - \sqrt{\beta_{3\theta}{}^2 - 3\gamma_{3\theta}})$$

$$= +\frac{1}{3}(1.6 - \sqrt{1.6^2 - 3(-3)}) = -0.6$$

The data from which *Figure 41* was created is displayed in *Table 33*. It locates an ***RST Spread*** which is numerically equivalent to that of the root structure z_R, z_S, and z_T inherent in the *given Generalized Cubic Function* $z^3 + 1.4z^2 - 3.2z - 0.84 = y_{TRANSFORMED}$. This is validated as follows:

First, it is verified that the *given Generalized Cubic Function* contains three real roots, where:

$$u^3 + 1.4u^2 - 3.2u - 0.84 = 0$$

$$V = \frac{B}{3} = \frac{1.4}{3} = 0.46666667$$

$$f = \sqrt{V^2 - \frac{C}{3}} = \sqrt{(0.46666667)^2 + \frac{3.2}{3}} = \sqrt{1.284444448} = 1.133333335$$

$$\psi = \frac{V^3 - 3f^2V - D}{2f^3}$$

$$= \frac{(0.46666667)^3 - 3(1.133333335)^2(0.46666667) + 0.84}{2(1.133333335)^3}$$

$$= -0.294219422$$

$$= \cos(6\omega)$$

Since the *given Generalized Cubic Function* has 3 real roots.

Next, its roots are determined as follows:

$$6\varpi = +107.1107364^o; -107.1107364^o; +107.1107364^o + 360^o$$

$$= +107.1107364^o; 252.8892636^o; +467.1107364^o$$

$$2\varpi = \frac{+107.1107364^o}{3}; \frac{252.8892636^o}{3}; \frac{+467.1107364^o}{3}$$

$$2\varpi = 35.7035788^o; 84.2964212^o; 155.7035788^o$$

$$\cos 2\varpi = 0.812047076; 0.099381902; -0.911428978$$

$$2f\cos(2\omega) = (2f)[0.812047076; 0.099381902; -0.911428978]$$

$$= 2(1.133333335)[0.812047076; 0.099381902; -0.911428978]$$

$$\ell = +1.840640042; 0.225265644; -2.065905687$$

Where,

$$u = \ell - V$$
$$= [+1.840640042;0.225265644;-2.065905687] - (0.46666667)$$
$$= 1.373973372;-0.241401026;-2.532572357$$
$$= z_R, z_S, z_T$$

These three values are entered into the first column of *Table 33* and return respective y values equal to zero in the second column. Since the *associated Displaced Family Curve* engages values which are an absolute value z_B, to the right, then a value of 1.6 must be added to each root in order to obtain roots for this *Parent Function*, as follows:

$$[1.373973372;-0.241401026;-2.532572357] + 1.6 = 2.973973372;1.358598974;-0.932572357$$

For each of these three z values listed, the third column of *Table 33* gives respective y′ values equal to zero.

Both *Displaced Family Curves* not only are identical in shape, but also are *aligned* with their *relative high points* located upon the y-axis. Since the roots supplied above occur on the z-axis a distance of ν below the *relative high point* of the *Associated Displaced Family Curve*, there exists an ***RST Spread*** just like it the same distance of ν below the relative *high point* of the *associated 3θ Displaced Family Curve*. This ordinate is computed as:

$$-(\nu_{3\theta} - \nu) = -(3.768 - 1.54133333) = -2.22666666$$

This value of y''' appears in the fifth column of *Table 33* for each of the three roots of the *associated Displaced Family Curve* (i.e., where $y' = 0$).

Lastly, the *associated 3θ Cubic Curve* is displaced to the left of its *associated 3θ Displaced Family Curve* a distance equal to $z_{B-3\theta} = -0.6$. Hence, the ***RST Spread*** which is identical to the roots of the *given Cubic Curve* occur at the following points on the z-axis:

$$[2.973973372;1.358598974;-0.932572357] - 0.6 = 2.373973372;0.758598974;-1.532572357$$

Naturally, their respective y′′ values also must equal -2.22666666, which appears in the fourth column of *Table 33* next to these adjacent z values.

<u>In conclusion</u>, it has been demonstrated that the *root structure* for any *given Generalized Cubic Function* can be characterized, or reduplicated, by an associated ***RST Spread*** contained within a *3θ Cubic Function* of the same exact *curve shape*. Hence ***RST Spreads***, inherent within ***3θ Cubic Functions*** characterize the *root structures* for <u>all</u> *Generalized Cubic Functions*.

Table 33. Equivalent *3θ Cubic Function* RST Spread *for Given Cubic Function.*

	Given Generalized Cubic Function	Associated Displaced Family Function	Associated 3θ Cubic Function	Associated Displace 3θ Cubic Function
$z = z' = z'' = z'''$	$z^3 + 1.4z^2 - 3.2z - 0.84 = y$	$z'^3 - 3.4z'^2 + 3.768 = y'$	$z''^3 - 1.6z''^2 - 3z'' + \frac{1.6}{3} = y''$	$z'''^3 - 3.4z'''^2 + 1.54133333 = y'''$
4	72.76	13.368	26.93333333	11.14133333
3.5	47.985	4.993	13.30833333	2.766333333
3	29.16	0.168	4.133333333	-2.058666667
2.973973375	28.32896976	0	3.763544912	-2.226666666
2.5	15.535	-1.857	-1.341666667	-4.083666667
2.373973375	12.83245408	-2.014429127	-2.226666666	-4.241095794
2	6.36	-1.832	-3.866666667	-4.058666667
1.373973372	0	-0.056738796	-4.01528049	-2.283405463
1.358598976	-0.095719083	0	-3.988039471	-2.22666666
1	-1.64	1.368	-3.066666667	-0.858666667
0.758598976	-2.025302576	2.247946596	-2.226666666	0.021279929
0.5	-1.965	3.043	-1.241666667	0.816333333
0	-0.84	3.768	0.533333333	1.541333333
-0.241401026	0	3.555799338	1.150229769	1.329132671
-0.6	1.368	2.328	1.541333333	0.101333333
-0.93257235065913	2.550749251	0	1.128494509	-2.226666666
-1	2.76	-0.632	0.933333333	-2.858666667
-1.53257235065913	3.7528485	-7.81751747	-2.226666666	-10.04418414
-1.6	3.768	-9.032	-2.858666667	-11.25866667
-2	3.16	-17.832	-7.866666667	-20.05866667
-2.532572357	0	-34.28306077	-18.37494943	-36.50972743
-3	-5.64	-53.832	-31.86666667	-56.05866667
-3.5	-15.365	-80.757	-51.44166667	-82.98366667
-4	-29.64	-114.632	-77.06666667	-116.8586667

SECTION 16. *THE PERFECT CUBIC FUNCTION.*

Equation 47 cited below represents the *Perfect Cubic Function* simply because any value 'y' on this *Cubic Curve* always is equal to the *exact cube* of its respective 'z' value. Furthermore, such 'z' value, as expressed therein, always is equal to the exact, or perfect *cube root* of any respective value that 'y' may assume.

Equation 47. ***The Perfect Cubic Function.***

$$y = z^3$$

16.1. Singularity.

The *Perfect Cubic Function* applies to a *family of Cubic Curves* designated below which assume *singular curve shape* no matter what value of 'a' is applied:

$$y = (z \pm a)^3$$

This is demonstrated by expanding the above stated *Cubic Curve Set* as a *perfect cube*, in order to express it in *Generalized Cubic Function* format as follows:

$$\begin{aligned} y &= (z \pm a)^3 \\ &= z^3 \pm (3a)z^2 + (3a^2)z \pm a^3 \end{aligned}$$

$$y_{TRANSFORMED} = z^3 + \beta' z^2 + \gamma' z + \delta'$$ *[Ref. Section 14.2.1]*

Where its *Parent Curve* is determined as,

$$y' = z'^3 + \sigma z'^2 + \nu$$ *[Ref. Section 14.2.1]*

Such that,

$$\sigma = -\sqrt{\beta'^2 - 3\gamma'}$$

$$\nu = \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta']$$

Then,

$$\begin{aligned} \sigma &= -\sqrt{\beta'^2 - 3\gamma'} \\ &= -\sqrt{9a^2 - 9a^2} \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
\nu &= \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')\sqrt{\beta'^2 - 3\gamma'} - 9\beta'\gamma' + 27\delta'] \\
&= \frac{1}{27}[2\beta'^3 + (2\beta'^2 - 6\gamma')(-\sigma) - 9\beta'\gamma' + 27\delta'] \\
&= \frac{1}{27}[2(\pm 3a)^3 + (2\beta'^2 - 6\gamma')(0) - 9(\pm 3a)(3a^2) + 27(\pm a^3)] \\
&= \frac{1}{27}[2(\pm 3a)^3 - 9(\pm 3a)(3a^2) + 27(\pm a^3)] \\
&= \frac{27}{27}(a^3)(\pm 2 \mp 3 \pm 1) \\
&= (a^3)(\pm 3 \mp 3) \\
&= (a^3)(0) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
y' &= z'^3 + \sigma z'^2 + \nu \\
&= z'^3 + (0)z'^2 + 0
\end{aligned}
$$

Or,

$$y' = z'^3$$

As such, each individual *Cubic Family Curve Set* conforming to the equality $y = (z \pm a)^3$ is an *identically shaped step function* of the *Perfect Cubic Parent Function* $y' = z'^3$; the only difference being that each individual *Cubic Family Curve Set* is displaced either to the left or to the right of the *Perfect Cubic Parent Function* by a distance 'a'. This is easily evidenced because:

When,

$$y' = y$$

Since,

$$y' = z'^3$$

$$y = (z \pm a)^3$$

Substitution renders the following *step function* relationship between z' and z for each and every arbitrary value of $y' = y$,

$$z'^3 = (z \pm a)^3$$

$$z' = z \pm a$$

The *significance* of this digression is that a *given Generalized Cubic Function* may be easily resolved whenever its γ′ coefficient is equal in value to one-third of the square of its β′ coefficient value, determined as follows:

Where,

$$\beta' = \pm 3a$$

$$\beta'^2 = 9a^2$$

$$\frac{\beta'^2}{3} = 3a^2$$

$$= \gamma'$$

16.2. The *Perfect Cubic Function* Algorithm.

The *Generalized Cubic Function* depicted below applies to the specific condition when its γ′ coefficient is equal in value to one-third of the square of its β′ coefficient value:

$$y_{TRANSFORMED} = z^3 + \beta' z^2 + \gamma' z + \delta' \qquad \text{[Ref. Section 14.2]}$$

$$= z^3 + \beta' z^2 + \frac{\beta'^2}{3} z + \delta'$$

Setting $\beta' = \pm 3a$ gives,

$$y_{TRANSFORMED} = z^3 \pm (3a) z^2 + \frac{(\pm 3a)^2}{3} z + \delta'$$

$$= z^3 \pm (3a) z^2 + \frac{9a^2}{3} z \pm (a^3 - a^3) + \delta'$$

$$= (z^3 \pm 3az^2 + 3a^2 z \pm a^3) \mp a^3 + \delta'$$

$$= (z \pm a)^3 \mp a^3 + \delta'$$

Accordingly,

$$y_{TRANSFORMED} \pm a^3 - \delta' = (z \pm a)^3$$

$$\sqrt[3]{y_{TRANSFORMED} \pm a^3 - \delta'} = z \pm a$$

$$\sqrt[3]{y_{TRANSFORMED} \pm a^3 - \delta'} \mp a = z$$

Since the *Cubic Resolution approach* presented in *Section 13.2* also applies for the specific case when $\gamma' = \beta'^2/3$, it ***is analogous to*** the solution afforded above for $y_{TRANSFORMED} = 0$ as follows:

$$z = \sqrt[3]{y_{TRANSFORMED} \pm a^3 - \delta'} \mp a$$

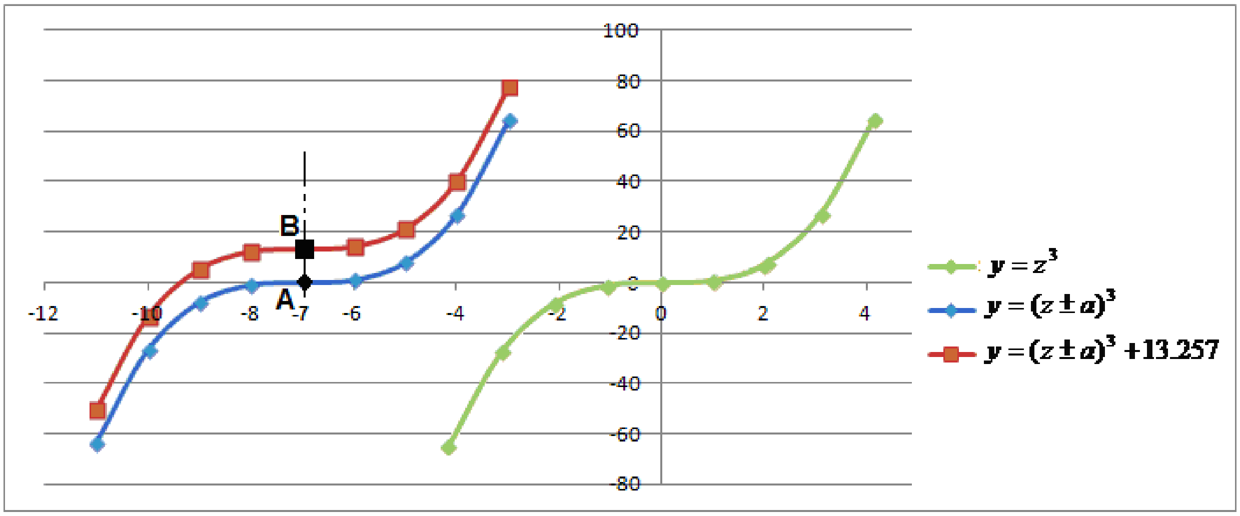
$$= \sqrt[3]{0 \pm (\beta/3)^3 - \delta'} \mp (\beta/3)$$

$$= 1/3(-\beta + \sqrt[3]{\beta^3 - 27\delta'})$$

16.3. Relativistic Interpretation.

The *Perfect Cubic Parent Function* is portrayed in *Figure 42*. It is accompanied by *two family curves* for the particular condition when $a = +7$ (alternatively denoted as $\pm a = \pm 7$; i.e., $+a = +7$ or $-a = -7$).

Figure 42. Perfect Cubic Parent Curve Plot along with Two Noted Family Curves.



z	y		
	z^3	$(z\pm a)^3$ $(z+7)^3$ $z^3+21z^2+147z+343$	$(z\pm a)^3+13.257$ $(z+7)^3+13.257$ $(z^3+21z^2+147z+343)+13.257$ $z^3+21z^2+147z+356.257$
6	216	2197	2210.257
5	125	1728	1741.257
4	64	1331	1344.257
3	27	1000	1013.257
2	8	729	742.257
1	1	512	525.257
0	0	343	356.257
-0.082331914	-0.00056	331.039	344.296
-1	-1	216	229.257
-2	-8	125	138.257
-3	-27	64	77.257
-4	-64	27	40.257
-5	-125	8	21.257
-6	-216	1	14.257
-7	-343	0	13.257
-8	-512	-1	12.257
-9	-729	-8	5.257
-10	-1000	-27	-13.743
-11	-1331	-64	-50.743
-12	-1728	-125	-111.743

Therein, the *Perfect Cubic Parent* Function crosses the z-axis at zero because:

$$y = z^3 \qquad \text{[Ref. Equation 47]}$$
$$0 = z^3$$
$$= (0)^3$$

The first *family curve* crosses the z-axis at -7 (Ref. *Point A*) determined as follows:

$$y = (z \pm a)^3$$
$$0 = (z \pm a)^3$$
$$0 = z \pm a$$
$$0 = z + 7$$

Or,

$$-7 = z$$

In other words, when $z = -7$, the first *family Cubic Curve* expressed in *Figure 42* crosses the z-axis, whereby $y = 0$.

Now, per the last equation depicted in the *Figure 42* legend, the second *family curve* is an exact duplicate of the first, except for the fact that its points reside a distance of 13.257 units above respective points illustrated in the *first family* curve.

Hence, when $z = -7$, y for the second *family curve* must reside at *point B* a distance of 13.257 above $y = 0$, shown *relativistically* as follows:

$$y = (z \pm a)^3 \qquad \text{[Ref. first family curve]}$$
$$0 = (-7 \pm a)^3$$
$$13.257 = (-7 \pm a)^3 + 13.257 \qquad \text{[Ref. second family curve]}$$

Since the second *family curve* is denoted as:

$$\text{y} = (z \pm a)^3 + 13.257$$
$$= [z^3 \pm (3a)z^2 + (3a^2)z \pm a^3] + 13.257$$
$$= [z^3 + (3)(7)z^2 + (3)(7)^2 z + 7^3] + 13.257$$
$$= (z^3 + 21z^2 + 147z + 343) + 13.257$$
$$= z^3 + 21z^2 + 147z + 356.257$$

All of these lines shown above are equivalent, or synonymous as indicated at the top of the last column of the *Figure 42* attached table.

16.4. Application.

The *Perfect Cubic Function Algorithm* applies only when the γ′ coefficient of any given *Generalized Cubic Function* is equal in value to one-third of the square of its β′ coefficient value.

The following analysis verifies that the following given *Generalized Cubic Function* qualifies for such resolution:

$$13.257 = z^3 + 21z^2 + 147z + 25.218$$

$$y_{TRANSFORMED} = z^3 + \beta' z^2 + \gamma' z + \delta'$$

$$\gamma' = 147(\frac{3}{3})$$

$$= \frac{441}{3}$$

$$= \frac{(21)^2}{3}$$

$$= \frac{\beta'^2}{3}$$

Then,

$$\beta' = 3a = 21$$

$$a = 7$$

$$a^3 = 7^3 = 343 \neq \delta'$$

$$z = \sqrt[3]{y_{TRANSFORMED} + a^3 - \delta'} - a$$

$$= \sqrt[3]{13.257 + 343 - 25.218} - 7$$

$$= \sqrt[3]{331.039} - 7$$

$$= 6.917668086 - 7$$

$$= -0.082331914$$

Check,

$$13.257 = z^3 + 21z^2 + 147z + 25.218$$

$$= (-0.082331914)^3 + 21(-0.082331914)^2 + 147(-0.082331914) + 25.218$$

$$= -0.00055809 + 0.1423494125 - 12.10279136 + 25.218$$

$$= -11.96100002 + 25.218$$

$$= 13.257 \qquad \text{Q.E.D.}$$

The given *Generalized Cubic Function* may be treated *relativistically* as follows:

$$13.257 = z^3 + 21z^2 + 147z + 25.218$$
$$13.257 = z^3 + 21z^2 + 147z + (343 - 343) + 25.218$$
$$13.257 = (z+7)^3 - 343 + 25.218$$
$$13.257 - (-343 + 25.218) = (z+7)^3$$
$$13.257 + 343 - 25.218 = (z+7)^3$$
$$331.039 = (z+7)^3$$

Or,

$$y = (z \pm a)^3$$

Since $a = \beta'/3 = 21/3 = 7$, the above equality is representative of the *first family Cubic Function* presented in *Figure 42*, and may be resolved as follows:

$$\sqrt[3]{331.039} = z + 7$$
$$6.917668086 = z + 7$$
$$6.917668086 - 7 = z$$
$$-0.082331914 = z$$

Q.E.D.

Notice that this value of $z = -0.082331914$ appears in the first column of the *Figure 42* attached table, whereby a value of $y = 331.039$ is rendered in its respective third column.

SECTION 17. *TAN θ TO ζ LINEARITY.*

Based solely upon a *specific manipulation* of *Characteristic Cubic Equation* coefficients (*Ref. Equation 31*), this section establishes a **new** significant *linear* relationship between $\zeta = \tan(3\theta)$ and its associated tan θ function.

17.1. Demonstrated Linearity.

Such linearity is represented by the following equation:

Equation 48. ***Tan θ to ζ Linearity Expression.***

$$\tan\theta = -(\frac{J}{F})\zeta$$

Where,

$$F = 2[3D - B]$$

$$J = 3(B + C) - (D + 1) \pm G$$

At,

$$G = \pm\sqrt{9(B^2 + C^2) + D^2 + 14BC - 6BD + 6CD + 1 + 6B - 6C - 34D}$$

As indicated, both "F" and "J" factors appearing in *Equation 48* represent combinations of the coefficients A, B, C, and D appearing in the *Characteristic Cubic Equation* (*ref. Equation 31*).

$$AR^3 + BR^2 + CR + D = 0 \qquad \textit{[Ref. Equation 31]}$$

Development of this formula is furnished below.

- First, the *Quadratic Formula* is applied to the *Simplified Unified Cubic Trigonometric Reduction Equation* as follows (Ref. *Equation 30*):

$$\zeta[C + 3D]\tan^2\theta - [B - 3D]\tan\theta - \zeta(D + 1) = 0 \qquad \textit{[Ref. Equation 30]}$$

Where,

$$\tan\theta = [-b \pm \sqrt{b^2 - 4ac}\,]/2a$$

$$= [(B - 3D) \pm \sqrt{(B - 3D)^2 + 4\zeta^2(3 + 3D)(D + 1)}\,]/2\zeta(3 + 3D)$$

- Squaring both sides of the above result then gives:

$$\tan^2\theta = \frac{(B - 3D)^2 \pm 2(B - 3D)\sqrt{(3D - B)^2 + 4\zeta^2(C + 3D)(D + 1)} + (3D - B)^2 + 4\zeta^2(C + 3D)(D + 1)}{2\zeta(C + 3D)2\zeta(C + 3D)}$$

- This relationship is equated to *Equation 37* as follows:

$$\tan^2\theta = \frac{(B-3D)^2 \pm 2(B-3D)\sqrt{(3D-B)^2+4\zeta^2(C+3D)(D+1)}+(3D-B)^2+4\zeta^2(C+3D)(D+1)}{2\zeta(C+3D)2\zeta(C+3D)} = \frac{3(B+C)+(D+1)\pm\sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}}{2(C+3D)}$$

$$\frac{(B-3D)^2 \pm 2(B-3D)\sqrt{(3D-B)^2+4\zeta^2(C+3D)(D+1)}+(3D-B)^2+4\zeta^2(C+3D)(D+1)}{2\zeta^2[C+3D]} = \frac{3(B+C)+(D+1)\pm\sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}}{1}$$

$$(B-3D)^2 \pm 2(B-3D)\sqrt{(3D-B)^2+4\zeta^2(C+3D)(D+1)}+(3D-B)^2+4\zeta^2(C+3D)(D+1) = 2\zeta^2(C+3D)\ [3(B+C)+(D+1)\pm\sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}]$$

At $(B-3D)^2=(3D-B)^2$

$$2(3D-B)^2 \pm 2(B-3D)\sqrt{(3D-B)^2+4\zeta^2(C+3D)(D+1)}+4\zeta^2(C+3D)(D+1) = 2\zeta^2(C+3D)\ [3(B+C)+(D+1)\pm\sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}]$$

- *Now at,*

$$E = C+3D$$

$$F = 2(3D-B)$$

$$G = \sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}$$

$$F(3D-B)\pm(-F)\sqrt{(3D-B)^2+4\zeta^2(E)(D+1)}+4\zeta^2(E)(D+1) = 2\zeta^2(E)\ [3(B+C)+(D+1)\pm G]$$

$$\pm(-F)\sqrt{(3D-B)^2+4\zeta^2(E)(D+1)}+4\zeta^2(E)(D+1) = 2\zeta^2(E)\ [3(B+C)+(D+1)\pm G]+F(B-3D)$$

$$\pm(-F)\sqrt{(3D-B)^2+4\zeta^2(E)(D+1)} = 2\zeta^2(E)\ [3(B+C)+(D+1)\pm G]+F(B-3D)-4\zeta^2(E)(D+1)$$

$$\pm(-F)\sqrt{(3D-B)^2+4\zeta^2(E)(D+1)} = 2\zeta^2(E)\ [3(B+C)+(D+1)-2(D+1)\pm G]+F(B-3D)$$

$$\pm(-F)\sqrt{(3D-B)^2+4\zeta^2(E)(D+1)} = 2\zeta^2(E)\ [3(B+C)-(D+1)\pm G]+F(B-3D)$$

- At $J = 3(B+C) - (D+1) \pm G$

$$\pm(-F)\sqrt{(3D-B)^2 + 4\zeta^2(E)(D+1)} = 2\zeta^2(E)\ (J) + F(B-3D)$$

$$\pm\sqrt{(3D-B)^2 + 4\zeta^2(E)(D+1)} = (3D-B) - \frac{2\zeta^2 EJ}{F}$$

- Squaring both sides renders,

$$(3D-B)^2 + 4\zeta^2(E)(D+1) = (3D-B)^2 - 2(3D-B)(\frac{2\zeta^2 EJ}{F}) + (\frac{2\zeta^2 EJ}{F})^2$$

$$= (3D-B)^2 - (3D-B)(\frac{4\zeta^2 EJ}{F}) + [\frac{4\zeta^4 (EJ)^2}{F^2}]$$

- Subtracting $(3D-B)^2$ from both sides of the equation gives,

$$4\zeta^2(E)(D+1) = -(3D-B)(\frac{4\zeta^2 EJ}{F}) + [\frac{4\zeta^4 (EJ)^2}{F^2}]$$

$$E(D+1) = -(3D-B)(\frac{EJ}{F}) + (\frac{\zeta EJ}{F})^2$$

$$EF^2(D+1) + (3D-B)EFJ = \zeta^2(EJ)^2$$

$$\frac{EF^2(D+1) + (3D-B)EFJ}{E(EJ^2)} = \zeta^2$$

$$\frac{F^2(D+1) + (3D-B)FJ}{EJ^2} = \zeta^2$$

$$\frac{F^2(D+1)\frac{2}{2} + \frac{F}{2}FJ}{\frac{2}{2}EJ^2} = \zeta^2$$

$$\frac{2F^2(D+1) + F^2 J}{2EJ^2} = \zeta^2$$

$$\frac{F^2[2(D+1) + J]}{2EJ^2} = \zeta^2$$

$$\frac{F^2[2(D+1) + 3(B+C) - (D+1) \pm G]}{2EJ^2} = \zeta^2$$

$$\frac{F^2[3(B+C) + (D+1) \pm G]}{2EJ^2} = \zeta^2$$

- Now, where:

$$\tan^2\theta = \frac{[3(B+C)+(D+1)] \pm \sqrt{9(B^2+C^2)+D^2+14BC-6BD+6CD+1+6B-6C-34D}}{2C+6D}$$ *[Ref. Equation 37]*

$$= \frac{[3(B+C)+(D+1)] \pm G}{2(C+3D)}$$

$$= \frac{[3(B+C)+(D+1)] \pm G}{2E}$$

- Substituting $\tan^2\theta$ into the derived relationship above yields:

$$\frac{F^2[3(B+C)+(D+1) \pm G]}{2EJ^2} = \zeta^2$$

$$\frac{F^2 \tan^2\theta}{J^2} = \zeta^2$$

- Lastly, taking the square root of both sides produces the following relationship:

$$\pm\frac{F\tan\theta}{J} = \zeta$$

- Selecting the negative result and re-arranging terms finally gives:

$$\tan\theta = -(\frac{J}{F})\zeta$$ *Q.E.D.*

17.2. Development of the *J-Function Cubic Expression*.

The above linearity relationship gives rise to yet another cubic expression by setting $\zeta = \frac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$ as follows:

$$\tan^3\theta = 3\tan\theta - \zeta(1-3\tan^2\theta)$$ *[Ref. Equation 3]*

$$-3\tan\theta + \tan^3\theta = -\zeta(1-3\tan^2\theta)$$

$$3\tan\theta - \tan^3\theta = \zeta(1-3\tan^2\theta)$$

$$\frac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta} = \zeta$$

$$\tan\theta = -(\frac{J}{F})\zeta$$ *[Ref. Equation 48]*

$$\tan\theta = -(\frac{J}{F})\frac{(\tan\theta)(3-\tan^2\theta)}{(1-3\tan^2\theta)}$$

Or,

$$1 = -(\frac{J}{F})\frac{(3-\tan^2\theta)}{(1-3\tan^2\theta)}$$

Cross multiplying yields:

$$F(1-3\tan^2\theta) = J(\tan^2\theta - 3)$$

Collecting like terms produces:

$$F + 3J = \tan^2\theta(3F + J)$$

Then,

$$\frac{F+3J}{3F+J} = \tan^2\theta$$

But,

$$\tan^2\theta = (\frac{J\zeta}{F})^2$$

$$= \frac{F+3J}{3F+J}$$

So, by equating terms which equal $\tan^2\theta$ above, the following *cubic relationship* unfolds:

$$(\frac{J\zeta}{F})^2 = \frac{F+3J}{3F+J}$$

Again, by cross multiplying:

$$J^2\zeta^2(3F+J) = F^2(F+3J)$$

$$J^2(3F+J) = (\frac{F}{\xi})^2(F+3J)$$

$$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0$$

This above equation is an expression which relates the *cubic function, J* to its associated $\zeta = \tan(3\theta)$ via the variable *F* as follows:

Equation 49. *J-function Cubic Expression.*

$$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0$$

Naturally, *Equation 49* may be **resolved** easily since its cubic variable "J' may be ascertained as follows:

$$\tan\theta = -(\frac{J}{F})\zeta \qquad \textit{[Ref. Equation 48]}$$

$$-(\frac{F}{\zeta})\tan\theta = J$$

So,

- For any desired value of "ζ", its associated "tan θ" is calculated
- Then, as an arbitrary value for "F" becomes introduced, its associated cubic variable "J' is readily determined.

Accordingly, *Equation 49* may be used *to resolve* any *Generalized Cubic Equation* whose coefficients match its own, as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0$$

By equating coefficients, the following relationships are established:

- $3F = \beta$

  $F = \frac{\beta}{3}$
- $-3(\frac{F}{\xi})^2 = \gamma$

  $\frac{-3F^2}{\gamma} = \xi^2$

  $F\sqrt{\frac{-3}{\gamma}} - = \xi$
- $-F(\frac{F}{\xi})^2 = \delta$

In order for these three above relationships to apply, it is necessary that "γ" be negative; furthermore, the "δ" term of the *Generalized Cubic Equation* under evaluation must be equal to the cube of the "F" value established earlier, divided by the square of its previously established "ζ".

As a brief example of this, consider the following given *Generalized Cubic Equation*:

$$z^3 + 15z^2 - 75z - 125 = 0$$

Since, the third, or "γ" coefficient of the equation is negative, the above equation *may* satisfy the conditions to allow resolution as a *J-Function Cubic Expression*. To determine whether this is possible, the following analysis is conducted:

$$F = \frac{\beta}{3}$$
$$= \frac{15}{3}$$
$$= 5$$

$$F\sqrt{\frac{-3}{\gamma}} = \xi$$

$$5\sqrt{\frac{-3}{-75}} = \zeta$$

$$5\sqrt{\frac{-3}{(25)(-3)}} = \zeta$$

$$\frac{5}{5}\sqrt{\frac{-3}{(-3)}} = \zeta$$

$$1 = \zeta$$

$$-F(\frac{F}{\xi})^2 = \delta$$

$$-5(\frac{5}{1})^2 = \delta$$

$$-125 = \delta$$

Since the "δ" term of the intended *Generalized Cubic Equation* does turn out to be equal to -125, then it *qualifies* for cubic resolution via invoking the *J-Function Cubic Expression* as follows:

$$\zeta = \tan(3\theta) = 1$$

$$3\theta = 45^o$$

$$\theta = 15^o$$

$$\tan\theta = 0.267949192$$

$$-(\frac{F}{\zeta})\tan\theta = J$$

$$-(\frac{5}{1})0.267949192 = J$$

$$-1.339745962 = J = z$$

Check,

$$z^3 + 15z^2 - 75z - 125 = 0$$
$$(-1.339745962)^3 + 15(-1.339745962)^2 - 75(-1.339745962) - 125 = 0$$
$$-2.404735808 + 26.92378865 + 100.4809472 - 125 = 0$$
$$0 = 0 \qquad \text{Q.E.D.}$$

The other two roots for the above stated *Generalized Cubic Equation* are determined as follows:

$$(z - z_R)(z^2 + Mz + N) = 0$$
$$z^3 + (M - z_R)z^2 + (N - Mz_R)z - Nz_R = 0$$
$$z^3 + 15z^2 - 75z - 125 = 0$$

$$-Nz_R = -125$$
$$N = \frac{125}{z_R}$$
$$= \frac{125}{-1.339745962}$$
$$= -93.30127019$$

$$M - z_R = 15$$
$$M = 15 + z_R$$
$$= 15 - 1.339745962$$
$$= 13.66025404$$

Where,

$$z^2 + Mz + N = 0$$

Via *Quadratic Formula*:

$$z_S, z_T = [-M \pm \sqrt{M^2 - 4(1)N}\,]/2(1)$$
$$= [-13.66025404 \pm \sqrt{(13.66025404)^2 + 4(1)(93.30127019)}\,]/2$$
$$= 5, -18.66025404$$

Check,

$$z^3 + 15z^2 - 75z - 125 = 0$$
$$5^3 + 15(5)^2 - 75(5) - 125 = 0$$
$$125 + 375 - 375 - 125 = 0$$
$$0 = 0 \qquad \text{Q.E.D.}$$

$$z^3 + 15z^2 - 75z - 125 = 0$$
$$(-18.66025404)^3 + 15(-18.66025404)^2 - 75(-18.66025404) - 125 = 0$$
$$-6497.595266 + 5223.076273 + 1399.519053 - 125 = 0$$
$$0 = 0 \qquad \text{Q.E.D.}$$

Secondly, now consider the following *Generalized Cubic Equation*:

$$z^3 + 15z^2 - 75z - 150 = 0$$

Since this equation merits the same exact results for calculations of $F = 5$ and $\zeta = 1$ as the one above:

$$-F(\frac{F}{\xi})^2 = -5(\frac{5}{1})^2$$

$$-125 \neq \delta = -150$$

Hence the *J-Function Cubic Expression* may not be applied in order to determine the unknown cubic variable "z".

However, the *Generalized Cubic Curve* $z^3 + 15z^2 - 75z - 150 = y$ is exactly the same *Cubic Function* as $z^3 + 15z^2 - 75z - 125 = y$, with the only exception being that its y values are displaced a distance of twenty-five units below respective y values on the $z^3 + 15z^2 - 75z - 125 = y$ curve.

Hence, *Cubic Equation roots* for the $z^3 + 15z^2 - 75z - 150 = 0$ *Generalized Cubic Equation* may be represented by the *associated* ***RST Spread*** on the $z^3 + 15z^2 - 75z - 125 = y$ curve which is located a distance of 25 units below the respective roots previously calculated for the $z^3 + 15z^2 - 75z - 125 = 0$ *Generalized Cubic Equation*.

17.3. *J-Function Cubic Expression* Equates to *3θ Cubic Function.*

The *J-Function Cubic Expression* equates to the *3θ Cubic Function* as follows:

$$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0 \qquad \textit{[Ref. Equation 49]}$$

Multiplying thru by $-(\zeta/F)^3$ gives,

$$-(\frac{\zeta}{F})^3 J^3 - 3\zeta(\frac{\zeta}{F})^2 J^2 + 3(\frac{\zeta}{F})J + \zeta = 0$$

Where,

$$-(F/\zeta)\tan\theta = J \qquad \textit{[Ref. Equation 48]}$$

Then, via substitution:

$$-(\frac{\zeta}{F})^3[(-F/\zeta)\tan\theta]^3 - 3\zeta(\frac{\zeta}{F})^2[(-F/\zeta)\tan\theta]^2 + 3(\frac{\zeta}{F})[(-F/\zeta)\tan\theta] + \zeta = 0$$

$$\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta = 0$$

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

17.4. Application.

The newly developed *Equations 48* and *Equation 49* may be applied many ways. Perhaps one of the most intriguing applications pertains to ascertaining respective values of ζ with respect to the tan θ for specific ratios of the two. Below is an example of how these values are determined for the specific condition when ratio of ζ/(tan θ) is equal to -4/3:

Where,

$$\tan\theta = -(\frac{J}{F})\zeta \qquad \textit{[Ref. Equation 48]}$$

$$-\frac{F}{J} = \frac{\zeta}{\tan\theta} = -\frac{4}{3}$$

Since *many* possible selections for a value of "F" exist, this analysis will start with the easiest to assess mathematically by assigning a value for "F" equal to 4 units. From the ratios afforded in the equation above, it follows then that "J" is equal to 3 units. Substituting these values into *Equation 49* renders the following expression:

$$J^3 + (3F)J^2 - 3(\frac{F}{\xi})^2 J - F(\frac{F}{\xi})^2 = 0 \qquad \textit{[Ref. Equation 49]}$$

$$(3)^3 + (3)(4)(3)^2 - 3(\frac{4}{\xi})^2(3) - 4(\frac{4}{\xi})^2 = 0$$

$$27 + 108 = (9+4)(\frac{4}{\xi})^2$$

$$135\zeta^2 = (13)(16)$$

$$\zeta = \pm\sqrt{\frac{(13)(16)}{135}}$$

$$= \pm 1.241265782$$

The negative value 'ζ', as determined above is applied as follows:

$$\tan(3\theta) = -1.241265782$$

$$3\theta = 128.8559344^o$$

$$\theta = 42.95197812^o$$

$$\tan\theta = +0.930949336$$

Hence,

$$\frac{\zeta}{\tan\theta} = \frac{-1.241265782}{+0.930949336}$$

$$= -\frac{4}{3}$$ Q.E.D.

Alternatively, this assessment can be conducted as follows, with the only difference being that it first solves for tan θ, instead of ζ.

Where,

$$\tan^3\theta = 3\tan\theta - \zeta(1 - 3\tan^2\theta)$$ *[Ref. Equation 3]*

$$\frac{\tan^3\theta - 3\tan\theta}{1 - 3\tan^2\theta} = -\zeta$$

$$\frac{\tan\theta(3 - \tan^2\theta)}{1 - 3\tan^2\theta} = \zeta$$

$$\frac{3 - \tan^2\theta}{1 - 3\tan^2\theta} = \frac{\zeta}{\tan\theta}$$

$$= -\frac{4}{3}$$

Cross multiplying yields:

$$3(3 - \tan^2\theta) = 4(3\tan^2\theta - 1)$$

Or,

$$9 - 3\tan^2\theta = 12\tan^2\theta - 4$$

$$13 = 15\tan^2\theta$$

$$\sqrt{\frac{13}{15}} = \tan\theta$$

Then,

$$0.930949336 = \tan\theta$$

SECTION 18. *GEOMETRIC CONSTRUCTION USING OTHER THAN 3θ DATA.*

Along with *Equation 36*, other *equation formats* now may be *specified* which link *trigonometric values* of an angle to those of one-third its size.

Inherent *coefficient structures* provide a pathway for *geometric construction* which associates such two trigonometric entities.

This is not the same thing as performing a *Euclidean trisection* because *certain independent information* which is contained within *such coefficient structures also* needs to be assessed, *in addition to* that which is *directly associated* with *a given angle 3θ*.

Three equation type *categories* are afforded below which encompass variations in *coefficient structures*:

1) Those comprised solely of *rationally-based coefficients (Ref. Section 9.1)*;
2) Those comprised solely of *cubic irrational coefficients*, or those which are not *rationally-based (Ref. Section 9.1)*; and
3) Those which contain a *combination* of *coefficients* fitting *Category 1* and *Category 2* descriptions.

A brief list of ***salient equation formats*** which can be *portrayed* and, thereby *further characterized* by *such* ***geometric construction*** is as follows:

- *An Equation 1 Reduction (Ref. Equation 4)*
- The *SUCTRE – A Quadratic Equation (Ref. Equation 30)*
- *The Tan θ to ζ Linearity Expression. (Ref. Equation 48)*
- Equations resulting when $z_R = -1/\tan(3\theta) = -1/\zeta$
- *Complex Quadratic Equations for the Angle Trisector Triangle*
- Equations emulated by the *Cosine Circle*

18.1. An *Equation 1* Reduction.

Complex Quadratic Equation 4, reiterated below, qualifies principally as a *Category 2 equation type*. When considering $\cos\theta$ as a non *rationally-based* quantity, only in very rare instances when the ratio of τ with respect to λ or $(2\tau\lambda - 5)/6\lambda$ become *rationally-based* does this *equation type* revert to *Category 3*.

Equation 4 correlates an unknown *quadratic quantity*, $\cos\theta$, to a given, or assigned value of $\tau = \cos(3\theta)$ via single *irrational association*, $\lambda = \sin(3\phi)$, rather than by an *irrational number set*.

$$\cos^2\theta + (\frac{2\tau\lambda - 5}{6\lambda})\cos\theta - \frac{\tau}{2\lambda} = 0$$ *[Ref. Equation 4]*

This is confirmed for the case when 3θ is equal to 60° below:

$$\tau = \cos(3\theta) = \cos 60^o = 1/2$$

$$3\theta = 60^o$$
$$\theta = 60^o / 3$$
$$= 20^o$$

$$\sin\phi = \frac{1}{2\cos\theta} = \frac{1}{2\cos 20^o} = 0.532088886$$

$$\phi = 32.1467014^o$$

$$3\phi = 96.44010419^o$$

$$\lambda = \sin(3\phi) = 0.993689653$$

Then,

$$\cos^2\theta + (\frac{2\tau\lambda - 5}{6\lambda})\cos\theta - \frac{\tau}{2\lambda} = 0$$

$$(\cos 20^o)^2 + (\frac{2\tau\lambda - 5}{6\lambda})\cos 20^o - \frac{\tau}{2\lambda} = 0$$

$$(\cos 20^\circ)^2 + [\frac{2(1/2)(0.993689653) - 5}{6(0.993689653)}](\cos 20^\circ) - \frac{(1/2)}{2(0.993689653)} = 0$$

$$\cos^2 20^\circ - 0.671958683\cos 20^\circ - 0.251587605 = 0$$

$$0.883022221 - 0.631434616 - 0.251587605 = 0$$

$$0 = 0$$

<u>In conclusion</u>, *Equation 1* <u>can</u> *be reduced further*. Although *present day conjecture* is that such equation is ***irreducible***, reduction becomes precipitated simply by supplying applicable *irrational coefficients*, as determined by mathematical calculation. As indicated, $\cos\theta$ may be determined directly via the *Euclidean mapping process* specified in Section 2.3, in order to enable its determination via *straightedge* and *compass*.

18.2. The SUCTRE.

The *Simplified Unified Cubic Trigonometric Reduction Equation (Ref. Equation 30)* is further examined since its *coefficients* merely reflect further *breakdowns* of coefficients 'a', 'b', and 'c' specified in the *Quadratic Equation* as follows:

For,

$$\zeta(C+3D)\tan^2\theta-(B-3D)\tan\theta-\zeta(D+1)=0 \qquad \textit{[Ref. Equation 30]}$$

Where,

$$a=\zeta(C+3D)$$
$$b=-(B-3D)$$
$$c=-\zeta(D+1)$$

$$ax^2+bx+c=0$$

Depending upon how its *coefficients* are depicted, the *SUCTRE* may fall into any of the three *categories* expressed above.

First, a *Category 1 SUCTRE* is determined for a given value of $\zeta=\tan(3\theta)=13/9$. This is achieved by calculating a *Characteristic Cubic Equation* from the *3θ Cubic Equation* shown below, and then ascertaining its associated SUCTRE.

$$z^3+\beta z^2+\gamma z+\delta=0$$
$$z^3-3\zeta z^2-3z+\zeta=0$$
$$z^3-3(\frac{13}{9})z^2-3z+\frac{13}{9}=0$$
$$z^3-(\frac{39}{9})z^2-3z+\frac{13}{9}=0$$

As indicated, the *resulting transformation* contains only *rational coefficients*, each of which is comprised of manipulations of the *rational numbers* expressed in the *coefficients* of the *3θ Cubic Equation* above, and its roots, demonstrated as follows:

$$\zeta=\tan(3\theta)=13/9$$

$$3\theta=55.30484647^o$$
$$\theta=18.43494882^o$$

$$\tan\theta=\frac{1}{3}$$

$$B = \frac{\beta}{\tan\theta} = -\frac{39}{9(1/3)} = -\frac{39}{3} = -13$$

$$C = \frac{\gamma}{\tan^2\theta} = -\frac{3}{(1/9)} = -27$$

$$D = \frac{\delta}{\tan^3\theta} = \frac{13}{9(1/27)} = 13(3) = 39$$

$$AR^3 + BR^2 + CR + D = 0$$ [Ref. Equation 31]

$$R^3 - 13R^2 - 27R + 39 = 0$$

For $R = 1$:

$$1 - 13 - 27 + 39 = 0$$

$$40 - 40 = 0$$

Then,

$$a = \zeta(C + 3D) = \frac{13}{9}[-27 + 3(39)] = 130$$

$$b = -(B - 3D) = -(-13 - 117) = 130$$

$$c = -\zeta(D+1) = -\frac{13}{9}(39+1) = -\frac{520}{9}$$

Hence, the resulting *SUCTRE*, exhibiting only *rational coefficients* is:

$$130x^2 + 130x - \frac{520}{9} = 0$$

A value for $x = \tan\theta$ is determined using the *Section 2.3 Euclidean mapping process*, as represented by the following *Quadratic Formula*:

$$x_{1;2} = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(130)}[-130 \pm \sqrt{(130)^2 + 4(130)(\frac{520}{9})}]$$

$$= \frac{1}{260}[-130 \pm \sqrt{(130)^2 + 4(130)(\frac{520}{9})}]$$

$$= \frac{1}{260}(-130 \pm \sqrt{46{,}944.444444})$$

$$= \frac{-130 \pm 216.66666667}{260}$$

$$= \frac{86.66666667}{260}; \frac{-346.66666667}{260}$$

$$= 1/3; -4/3$$

Check,

At $\tan\theta = \dfrac{1}{3}$:

$$130x^2 + 130x - \frac{520}{9} = 0$$

$$130(1/3)^2 + 130(1/3) - \frac{520}{9} = 0$$

$$\frac{130}{9} + \frac{130(3)}{9} - \frac{520}{9} = 0$$

$$\frac{130(4)}{9} - \frac{130(4)}{9} = 0$$

$$0 = 0$$

At $\tan\theta = -\dfrac{4}{3}$

$$130(-4/3)^2 - 130(4/3) - \frac{520}{9} = 0$$

$$\frac{130(16)}{9} - \frac{130(4)}{3}(\frac{3}{3}) - \frac{130(4)}{9} = 0$$

$$\frac{130(16-12-4)}{9} = 0$$

$$\frac{130(16-16)}{9} = 0$$

$$0 = 0$$

In the above analysis, both $\zeta = \tan(3\theta)$ and $\tan\theta$ are determined to be *rational*. This is <u>highly unusual</u> because in most circumstances when $\zeta = \tan(3\theta)$ is *rational*, the associated $\tan\theta$ is *irrational*.

<u>Second</u>, a *Category 2 equation type SUCTRE* example is determined for $\zeta = \tan(3\theta) = \tan 85^o = 11.4300523$. Shown below is the *Generalized Cubic Equation*, otherwise portrayed as a *3θ Cubic Equation*:

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

$$z^3 - 3(11.4300523)z^2 - 3z + 11.4300523 = 0$$

$$z^3 - 34.29015691z^2 - 3z + 11.4300523 = 0$$

As indicated, the *resulting transformation* contains only *irrational coefficients*, each of which is comprised of manipulations of its *irrational* $\tan\theta$ root, demonstrated as follows:

$$\theta = 85^o / 3 = 28.333333^o$$

$$\tan\theta = 0.539195205$$

$$B = \frac{\beta}{\tan\theta} = -\frac{34.29015691}{0.53919205} = -63.59507017$$

$$C = \frac{\gamma}{\tan^2\theta} = -\frac{3}{(0.53919205)^2} = -10.31880039$$

$$D = \frac{\delta}{\tan^3\theta} = \frac{11.4300523}{(0.53919205)^3} = 72.91387054$$

$$AR^3 + BR^2 + CR + D = 0$$ `[Ref. Equation 31]`

$$R^3 - 63.59507017R^2 - 10.31880039R + 72.91387054 = 0$$

For $R = 1$:

$$1 - 63.59507017 - 10.31880039 + 72.91387054 = 0$$

$$73.91387054 - 73.91387054 = 0$$

Then,

$$a = \zeta(C + 3D) = 11.4300523[-10.31880039 + 3(72.91387054)] = 2382.283634$$

$$b = -(B - 3D) = -(-63.59507017 - 218.7416116) = 282.3366818$$

$$c = -\zeta(D + 1) = -11.4300523(72.91387054 + 1) = -844.8394062$$

Hence, the resulting *SUCTRE*, exhibiting only *irrational coefficients* is:

$$2382.283634x^2 + 282.3366818x - 844.8394062 = 0$$

A value for $x = \tan\theta$ is determined using the Section 2.3 *Euclidean mapping process*, as represented by the following *Quadratic Formula*:

$$x_{1;2} = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$$

$$= \frac{1}{2(2382.283634)}[-282.3366818 \pm \sqrt{(282.3366818)^2 + 4(2382.283634)(844.8394062)}]$$

$$= \frac{-282.3366818 \pm \sqrt{8{,}130{,}302.365}}{4764.567268}$$

$$= \frac{-282.3366818 \pm 2851.368507}{4764.567268}$$

$$= 0.539195205; -0.657710346$$

$$= \tan\theta; -\frac{1}{\tan(2\theta)}$$ *[Ref. Section 14.1.2]*

Check,

$$b/a = 282.3366818/2382.283634 = -(x_1 + x_2)$$
$$= -(0.539195205 - 0.657710346)$$
$$= 0.118515141$$

$$c/a = -844.8394062/2382.283634 = x_1 x_2$$
$$= (0.539195205)(-0.657710346)$$
$$= 0.354634265$$

Third, a *Category 3 equation type SUCTRE* example is presented below where *its first coefficient* is equal to *unity*, and its *last two* are *irrational*.

$$x^2 - 1.379385242x + 0.43969262 = 0$$

Where,

$$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{1.379385242 \pm \sqrt{(1.379385242)^2 - 4(1)(0.43969262)}}{2(1)}$$

$$= \frac{1.379385242 \pm \sqrt{0.143933165}}{2}$$

$$= \frac{1.379385242 \pm 0.379385242}{2}$$

$$= +0.879385242; +\frac{1}{2}$$

$$= (2\cos 20^o - 1); +\frac{1}{2}$$

The *irrational structure* of the last two coefficients is validated as follows:

$$b/a = b/1 = b = -(x_1 + x_2)$$

$$= \frac{1}{2} - 2\cos 20^o$$

$$c/a = c/1 = c = x_1 x_2$$

$$= \cos 20^o - \frac{1}{2}$$

Where, both results are clearly *irrational numbers*.

Below, the *coefficient factors* of *Equation 30* are substituted into the *Quadratic Formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\tan\theta = \frac{(B - 3D) \pm \sqrt{(3D - B)^2 + 4\zeta^2 (C + 3D)(D + 1)}}{2\zeta(C + 3D)}$$

As indicated, $x = \tan\theta$ is easily computed, or constructed once $\zeta = \tan(3\theta)$ along with the associated coefficients 'A', 'B' and 'C' from the *Characteristic Cubic Equation (Ref. Equation 31)* become specified.

18.3. The Tan θ to ζ Linearity Expression.

Equation 48 generally depicts a *Category 2 equation type* because its *left-hand member* is usually *irrational*.

Because it maps out a *straight line* meeting the equation $y = mx + b$, the ***tan θ to ζ Linearity Expression***, shown below, represents the simplest of all *Euclidean constructions*:

$$\tan\theta = -(\frac{J}{F})\zeta \qquad \textit{[Ref. Equation 48]}$$

Whereby, the slope "m" may be represented as:

$$m = -(\frac{J}{F})$$

The y-intercept is:

$$b = 0$$

The $\tan\theta$ becomes the resulting *ordinate value* for any and all x-axis values of ζ which may be represented on a straight line of slope -J/F which passes through the origin.

18.4. Equations resulting when z_R = -1/tan (3θ) = -1/ζ.

Generalized Cubic Equations which express $\alpha = 1$ are either *Category 1* or *Category 3 equation types*, depending upon the nature of their *remaining coefficients*.

Such dichotomy is demonstrated below where <u>two</u> specific sets of *Generalized Cubic Equations* are afforded along with their associated *Quadratic Equation reductions*. *One set* exhibits only *coefficients* which are *rationally-based (Ref. Section 9.1)*, while the last two *coefficients* of the *other set* are determined to be *completely cubic irrational*. Both *sets of equations* are established by selecting a *specific value* of z_R as follows:

$$z_R = R\tan\theta = \tan\theta_R = -\frac{1}{\zeta} = -\frac{1}{\sqrt{3}}$$

$$\theta_R = \arctan(-\frac{1}{\zeta}) = \arctan(-\frac{1}{\sqrt{3}})$$

$$= -30^o$$

$$\zeta = \tan(3\theta) = \sqrt{3}$$

$$3\theta = \arctan\sqrt{3} = 60^o = \theta_R + \theta_S + \theta_T$$

$$= -30^o + \theta_S + \theta_T$$

$$90^o = \theta_S + \theta_T$$

Hence, θ_S and θ_T are *complementary* to one another where,

$$\tan\theta_S = \frac{1}{\tan\theta_T}$$

When θ_S is selected specifically as 45°, θ_T becomes 45° also, such that:

$$\tan 45^o = \frac{1}{\tan 45^o} = 1 = z_S = z_T$$

The associated *Generalized Cubic Equation* is determined as:

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$z^3 - (z_R + z_S + z_T)z^2 + (z_R z_S + z_R z_T + z_S z_T)z - z_R z_S z_T = 0$$

$$z^3 - (2 - \frac{1}{\zeta})z^2 + (1 - \frac{2}{\zeta})z + \frac{1}{\zeta} = 0$$

One of its associated *reduced Quadratic Equations* is:

$$(z - z_R)(z - z_S) = 0$$

$$z^2 - (z_R + z_S)z + z_R z_S = 0$$

$$z^2 - (1 - \frac{1}{\zeta})z - \frac{1}{\zeta} = 0$$

For the second *Generalized Cubic Equation set*, z_R remains equal to $-1/\zeta$, but $\theta_S \neq 45^o$, where:

$$z_S = \tan\theta_S = \frac{1}{\tan\theta_T}$$

$$z_T = \tan\theta_T = \frac{1}{\tan\theta_S}$$

This produces,

$$(z - z_R)(z - z_S)(z - z_T) = 0$$

$$z^3 - (z_R + z_S + z_T)z^2 + (z_R z_S + z_R z_T + z_S z_T)z - z_R z_S z_T = 0$$

$$z^3 - (-\frac{1}{\zeta} + \tan\theta_S + \frac{1}{\tan\theta_S})z^2 + [-(\frac{1}{\zeta})(\tan\theta_S + \frac{1}{\tan\theta_S}) + 1]z + \frac{1}{\zeta} = 0$$

And, one of its associated *reduced Quadratic Equations* becomes:

$$(z - z_R)(z - z_S) = 0$$

$$z^2 - (z_R + z_S)z + z_R z_S = 0$$

$$z^2 - (\tan\theta_S - \frac{1}{\zeta})z - \frac{\tan\theta_S}{\zeta} = 0$$

Thereafter, a *cubic irrational* value $\tan\theta_S = \tan 20^o = 0.363970234$ is ascribed to the *second set* of equations.

The resulting two *equation type sets* are postulated below:

Category 1 Equation Type Sets	***Category 3 Equation Type Sets***
$z^3 - (2 - \frac{1}{\zeta})z^2 + (1 - \frac{2}{\zeta})z + \frac{1}{\zeta} = 0$	$z^3 - (\tan\theta_S + \frac{1}{\tan\theta_S} - \frac{1}{\zeta})z^2 + [1 - (\frac{1}{\zeta})(\tan\theta_S + \frac{1}{\tan\theta_S})]z + \frac{1}{\zeta} = 0$
$z^3 - 1.422649731z^2 - 0.154700538z + 0.577350269 = 0$	$z^3 - 2.534097385z^2 - 0.79639514z - 0.577350269 = 0$
$z^2 - (1 - \frac{1}{\zeta})z - \frac{1}{\zeta} = 0$	$z^2 - (\tan\theta_S - \frac{1}{\zeta})z - \frac{\tan\theta_S}{\zeta} = 0$
$z^2 - 0.42264973z - 0.577350269 = 0$	$z^2 + 0.213380034z - 0.21038312 = 0$

Both the *Category 1* and *Category 3 reduced Quadratic Equations* appearing in the third row of the above table may be operated upon via the *Euclidean Quadratic Mapping* process set forth in Section 2.3.

This validates the fact that a *compass and straight edge operation* can be applied ***without reservation*** upon given *Quadratic Equations* whose *coefficients* are either purely *rationally-based* lengths, or a combination thereof.

That is because once ***cubic irrational lengths*** become specified as *Quadratic Equation coefficients*, their *respective roots* can be determined via *Euclidean constructions based upon* such *presented lengths*.

18.5. *Complex Quadratic Equations* for the *Angle Trisector Triangle*.

Angle Trisector Triangles feature included angles of $\alpha - \phi$ and $3\alpha + \phi$ under specific circumstances when:

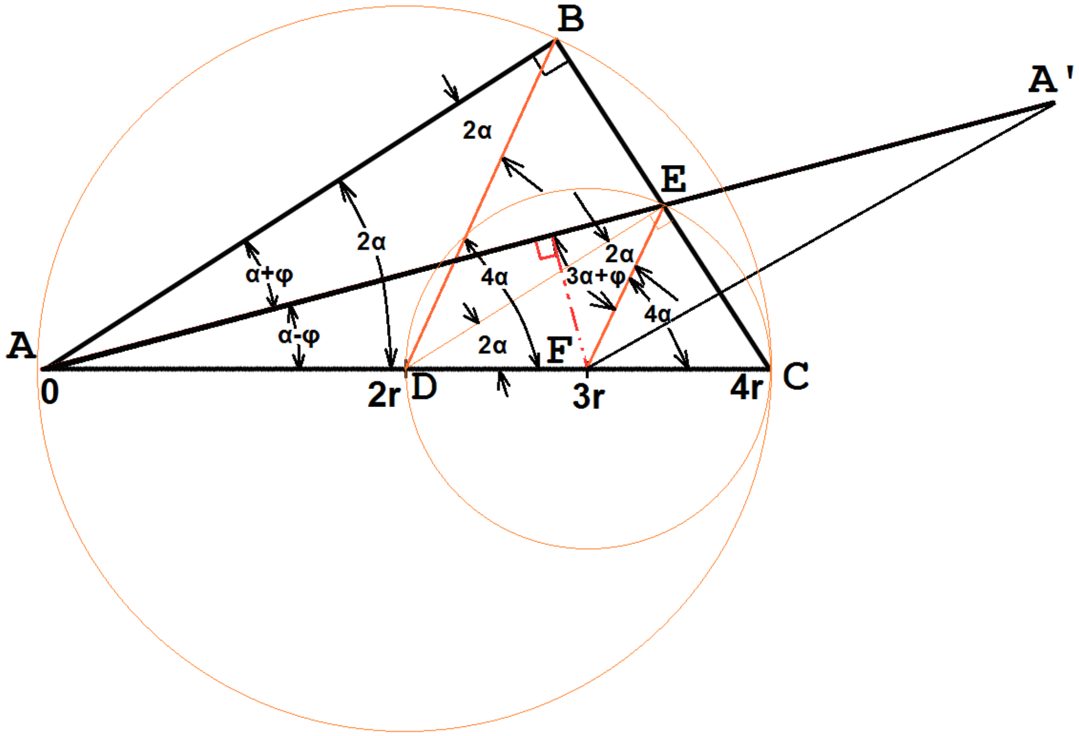
$$\tan\phi = \tan^3\alpha$$

They enable *(Ref. Figure 43, triangle AEF)*:

- Angles of $\alpha - \phi$ to be ***geometrically constructed*** from *given*, or *known* angles of $3\alpha + \phi$, thereby permitting a *geometric determination* of constituent angles α, 3α and ϕ
- *Rationally-based and cubic irrational lengths to coexist* within *single triangles (Ref. Section 9.1)*
- *Mathematical association* of *such lengths via Complex Quadratic Equation 50* derived below

Figure 43 depicts *right triangle ABC* inscribed inside a circle whose center lies at *Point D* and whose diameter is set equal to arbitrary length $\overline{AC} = 4r$. Once Point B is located upon the circumference of such circle, $\angle BAC$ is *arbitrarily identified* as magnitude 2α degrees. Since radius $\overline{DA} = \overline{DB} = 2r$, *triangle ABD* is isosceles, such that $\angle BAD = \angle ABD = 2\alpha$. Hence $\angle BDC$ is double this amount, or 4α.

Figure 43. Geometric Construction of Angle Trisector Triangles.



Next, *right triangle DEC* is inscribed inside a *second circle* whose diameter $\overline{DC}$ is set equal to the radius of the first circle, or length 2r. With its center residing at *Point F*, its radii $\overline{FD} = \overline{FC} = \overline{FE} = r$. Hence, *length* $\overline{AF} = 2r + r = 3r$. *Point E* is located where such *second circle* intersects line $\overline{BC}$.

The two *inscribed* ***right triangles*** are similar to one another since both contain a common angle $\angle \mathrm{ACB} = \angle \mathrm{DCE}$. Then, the *remaining included angles* $\angle \mathrm{BAC} = \angle EDC = 2\alpha$.

Since *triangle DEC* is one half the size of *similar triangle ABC*, with proportional respective sides, length $\overline{EC}$ is one-half length $\overline{BC}$. Straight line $\overline{AE}$ then may be designated as a *median*, thereby establishing ***Angle Trisector Triangle AEF***.

Once $\angle \mathrm{BAE}$ becomes designated at $\alpha + \phi$ degrees:

$$\begin{aligned}\angle \mathrm{EAC} &= \angle \mathrm{BAC} - \angle \mathrm{BAE} \\ &= 2\alpha - (\alpha + \phi) \\ &= \alpha - \phi\end{aligned}$$

With ΔDEF being isosceles:

$$\angle DEF = \angle EDF = 2\alpha$$

$$\begin{aligned} \angle EFC &= \angle EDF + \angle DEF \\ &= 2\alpha + 2\alpha \\ &= 4\alpha \end{aligned}$$

But,

$$\begin{aligned} \angle EFC = 4\alpha &= \angle EAF + \angle AEF \\ &= \angle EAC + \angle AEF \\ &= (\alpha - \phi) + \angle AEF \end{aligned}$$

$$3\alpha + \phi = \angle AEF$$

Angle Trisector Triangle AEF is easily identified because the length of one of its sides is equal to exactly *three times* that of its *adjacent side*.

Applying the *Law of Cosines* ***FOOTNOTE 1*** to triangle *AEF* gives the following:

Designating,

$$\overline{AE} = a$$

$$\overline{EF} = r$$

$$\overline{AF} = 3r$$

$$\overline{AF}^2 = \overline{AE}^2 + \overline{EF}^2 - 2(\overline{AE})(\overline{EF})\cos(3\alpha + \phi) \quad \textbf{\textit{FOOTNOTE 2}}$$

$$(3r)^2 = a^2 + r^2 - 2ar\cos(3\alpha + \phi)$$

$$9r^2 = a^2 + r^2 - 2ar\cos(3\alpha + \phi)$$

Re-arranging terms produces:

Equation 50. ***The Complex Quadratic Equation for the Angle Trisector Triangle.***

$$a^2 - 2ar\cos(3\alpha + \phi) - 8r^2 = 0$$

When $\cos(3\alpha + \phi)$ is considered to be a known quantity, *Complex Quadratic Equation 50* may be *arranged* in either of two ways as follows:

$$a^2 - [2r\cos(3\alpha + \phi)]a - 8r^2 = 0$$

$$r^2 + [\frac{a\cos(3\alpha + \phi)}{4}]r - \frac{a^2}{8} = 0$$

The *first arrangement* presents 'a' as the *unknown quantity*. Such variable may be determined with respect to given, or supplied, values of $\cos(3\alpha + \phi)$ and 'r' via the *Euclidean construction mapping approach* expressed in *Section 2.3.*

1. CRC Standard Mathematical Tables Twelfth Edition; The Chemical Rubber Co. Cleveland, OH; January 1964; page 410.
2. Ibid.

The *second arrangement* applies when ‘r’ is considered to be the *unknown quantity*. It again may be determined with respect to given values of $\cos(3\alpha+\phi)$ and ‘a’ by means of the same *Euclidean construction mapping approach* presented in *Section 2.3.*

The two *arrangements* offer *different, unique viewing perspectives*. One major difference in these is that the *unknown quantity* appearing in the *first arrangement* may be determined by the following *simplified construction*:

1) An angle of $3\alpha+\phi$ degrees is drawn whose one side $\overline{EF}=r$ and whose other side remains of indeterminate length;
2) A circle is drawn whose center resides and Point F and whose radius is equal to length 3r;
3) The intersection point between this circle and the aforementioned straight line of *indeterminate length* establishes $\overline{AE}=a$.

With respect to *Figure 43*:

a) When $\angle AEF=3\alpha+\phi$ is *acute*, $a=\overline{AE}$
b) When $\angle AEF=3\alpha+\phi$ is *obtuse*, $a=\overline{A'E}$

This is demonstrated by first constructing a perpendicular line segment from *Point F* to side $\overline{AE}$. An arc of length $\overline{FA}=3r$ then is swung. *Point A′* represents its intersection with the extended side $\overline{AE}$. Triangle A′EF then constitutes a *second* ***Angle Trisector Triangle*** since side $\overline{FA'}=3r$ and $\overline{EF}=r$.

The *significance* of the two ***Complex Quadratic, Angle Trisector Triangles*** is that an <u>exact trisector</u> for 3α may be constructed via *sole compass and straight edge operation* once supplied with either portion of the *number set* [$\overline{AE}=a$ and $\cos(3\alpha+\phi)$; or $\overline{EF}=r$ and $\cos(3\alpha+\phi)$].

With regard to *Equation 50*, any of the ‘a’, ‘r’, and $\cos(3\alpha+\phi)$ values may be either *rationally based* or *cubic irrational*. Once a *particular set* becomes chosen, or selected, it applies to <u>both</u> the *first arrangement* and *second arrangement*, since they exist merely as different *mathematical reflections* of one another. Such so-called <u>variable equivalency</u> is demonstrated in *two respective tables*, where the combinations shown depict *rationally based to cubic irrational pairings*, or *possibilities*.

In these *two tables* below, *Equation 50* is depicted either as a:

a) ***Category 1 equation type*** because when both quantities 'a' and 'r' are *rationally-based*, the *first* and *third terms (namely, "Ax^2" and "C")* also must be *rationally-based. [This holds with respect to* ***both*** *of the equations represented in the tables below!]* Hence, in order to ***preserve*** *equation equality*, the *second term*, *"Bx"*, also must remain *rationally-based*; or

b) ***Category 3 equation type*** when only 'a', 'r' or both quantities are *cubic irrational*, because:

- When only 'a' is *cubic irrational*, quantity 'r' must be *rationally-based*. As such, respective *first* and *third terms* must be *cubic irrational* and *rationally-based*; or vice-versa
- When only 'r' is *cubic irrational*, quantity 'a' must be *rationally-based*. As such, respective *first* and *third terms* must be *cubic irrational* and *rationally-based*; or vice-versa
- When *both* of these *quantities* are *cubic irrational*, coefficient ***A=1*** (see below) is *rationally-based* while coefficient C (depicted below as either $-8r^2$ or $-a^2/8$) must both be *cubic irrational*

r	cos(3α+ϕ)	COEFFICIENTS			EQUATION TYPE: $Ax^2+Bx+C=0$ ***For*** $x_{ABOVE}=a$: $a^2-[2r\cos(3\alpha+\phi)]a-8r^2=0$	a	EQN. CAT.
		A	B	C			
2 *(R-B)*	-23/12 *(Rat.-based)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Rat.-based)*	-8r² *(R-B)*	$a^2+\frac{23}{3}a-32=0$	3 *(Rat.-based)*	1
cos20° *(Cubic i.)*	3/4 *(Rat.-based)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	-8r² *(Cubic i.)*	$a^2-1.409538961a-7.064177772=0$	3.454474499 *(Cubic irrat.)*	3
cos20° *(Cubic i.)*	-10.807924 *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	-8r² *(Cubic i.)*	$a^2+20.31225391a-7.064177772=0$	sin20° *(Cubic irrat.)*	3
cos20° *(Cubic i.)*	1/ cos20° *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Rat.-based)*	-8r² *(Trans.)*	$a^2-2a-7.064177772=0$	3.839749597 *(Cubic irrat.)*	3
6 *(R-B)*	-70.142803 *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	-8r² *(R-B)*	$a^2+841.7136472a-288=0$	sin20° *(Cubic irrat.)*	3
sin20° *(Cubic i.)*	-4.71403069 *(Trans.)*	1 *(R-B)*	$-2r\cos(3\alpha+\phi)$ *(Cubic irrational)*	-8r² *(Cubic i.)*	$a^2+3.224586908a-0.935822227=0$	$2-\sqrt{3}$ *(Rat.-based)*	3

a	cos(3α+φ)	COEFFICIENTS			EQUATION TYPE: $Ax^2+Bx+C=0$ **For** $x_{ABOVE}=r$: $r^2+[\frac{a\cos(3\alpha+\phi)}{4}]r-\frac{a^2}{8}=0$	r	EQN. CAT.
		A	B	C			
3 *(Rat.-based)*	-23/12 *(Rat.-based)*	1 *(R-B)*	$\frac{a\cos(3\alpha+\phi)}{4}$ *(Rat.-based)*	$-a^2/8$ *(R-B)*	$r^2-\frac{23}{16}r-\frac{9}{8}=0$	2 *(R-B)*	1
3.454474499 *(Trans.)*	¾ *(Rat.-based)*	1 *(R-B)*	$\frac{a\cos(3\alpha+\phi)}{4}$ *(Cubic irrational)*	$-a^2/8$ *(Cubic i.)*	$r^2+0.647713968r-1.491674258=0$	$\cos 20^o$ *(Cubic i.)*	3
$\sin 20^o$ *(Trans.)*	-10.807924 *(Trans.)*	1 *(R-B)*	$\frac{a\cos(3\alpha+\phi)}{4}$ *(Cubic irrational)*	$-a^2/8$ *(Cubic i.)*	$r^2-0.924131976r-0.014622222=0$	$\cos 20^o$ *(Cubic i.)*	3
3.839749597 *(Trans.)*	$1/\cos 20^o$ *(Trans.)*	1 *(R-B)*	$\frac{a\cos(3\alpha+\phi)}{4}$ *(Cubic irrational)*	$-a^2/8$ *(Cubic i.)*	$r^2+1.021544043r-1.892959621=0$	$\cos 20^o$ *(Cubic i.)*	3
$\sin 20^o$ *(Trans.)*	-70.142803 *(Trans.)*	1 *(R-B)*	$\frac{a\cos(3\alpha+\phi)}{4}$ *(Cubic irrational)*	$-a^2/8$ *(Cubic i.)*	$r^2-5.997562963r-0.014622222=0$	6 *(R-B)*	3
$2-\sqrt{3}$ *(Rat.-based)*	-4.71403069 *(Trans.)*	1 *(R-B)*	$\frac{a\cos(3\alpha+\phi)}{4}$ *(Cubic irrational)*	$-a^2/8$ *(R-B)*	$r^2-0.315780179r+\frac{4\sqrt{3}-7}{8}=0$	$\sin 20^o$ *(Cubic i.)*	3

Notice *from the tables above* that when *both* quantities 'a' and 'r' are *cubic irrational*, the $\cos(3\alpha+\phi)$ can assume either a *rational*, or an *cubic irrational* value

In all *table depictions, coefficient* $A=1$. It could easily be changed to denote an *cubic irrational quantity* simply by dividing *each coefficient* contained in a given *Equation 50* by any *cubic irrational number.* As a transform to *Equation 50*, such new representation then could manifest itself as a *Category 2 equation type.*

Equation equality is *preserved* when the sum of the *three terms* in *Equation 50* equals zero. As demonstrated in the *two* additional tables shown below, this can be achieved only when either:

- All three terms are *rationally based*
- All three terms are *cubic irrational*
- Two of the terms are *cubic irrational* such that they sum to the value of a *rationally-based* third term

r	EQUATION TYPE: $Ax^2+Bx+C=0$ For $x_{ABOVE}=r$: $r^2+[\frac{a\cos(3\alpha+\phi)}{4}]r-\frac{a^2}{8}=0$	TERMS		
		Ax^2	**Bx**	**C**
2 *(Rat.-based)*	$r^2-\frac{23}{16}r-\frac{9}{8}=0$	4 *(Rat.-based)*	-23/8 *(Rat.-based)*	-9/8 *(Rat.-based)*
cos20° *(Cubic irrational)*	$r^2+0.647713968r-1.491674258=0$	0.883022221 *(Cubic irrational)*	0.608652026 *(Cubic irrational)*	-1.491674258 *(Cubic irrational)*
cos20° *(Cubic irrational)*	$r^2-0.924131976r-0.014622222=0$	0.883022221 *(Cubic irrational)*	-0.868399998 *(Cubic irrational)*	-0.014622222 *(Cubic irrational)*
cos20° *(Cubic irrational)*	$r^2+1.021544043r-1.892959621=0$	0.883022221 *(Cubic irrational)*	+0.959937399 *(Cubic irrational)*	-1.892959621 *(Cubic irrational)*
6 *(Rat.-based)*	$r^2-5.997562963r-0.014622222=0$	36 *(Rat.-based)*	-35.98537778 *(Cubic irrational)*	-0.014622222 *(Cubic irrational)*
sin20° *(Cubic irrational)*	$r^2-0.315780179r+\frac{4\sqrt{3}-7}{8}=0$	0.116977778 *(Cubic irrational)*	-0.108003182 *(Cubic irrational)*	-0.008974596 *(Rat.-based)*

a	EQUATION TYPE: $Ax^2+Bx+C=0$ For $x_{ABOVE}=a$: $a^2-[2r\cos(3\alpha+\phi)]a-8r^2=0$	TERMS		
		Ax^2	**Bx**	**C**
3 *(Rat.-based)*	$a^2+\frac{23}{3}a-32=0$	9 *(Rat.-based)*	23 *(Rat.-based)*	-32 *(Rat.-based)*
3.454474499 *(Cubic irrational)*	$a^2-1.409538961a-7.064177772=0$	11.93339406 *(Cubic irrational)*	-4.869216396 *(Cubic irrational)*	-7.064177772 *(Cubic irrational)*
sin20° *(Cubic irrational)*	$a^2+20.31225391a-7.064177772=0$	0.116977778 *(Cubic irrational)*	6.947199994 *(Cubic irrational)*	-7.064177772 *(Cubic irrational)*
3.839749597 *(Cubic irrational)*	$a^2-2a-7.064177772=0$	14.74367697 *(Cubic irrational)*	-7.679499194 *(Cubic irrational)*	-7.064177772 *(Cubic irrational)*
sin20° *(Cubic irrational)*	$a^2+841.7136472a-288=0$	0.116977778 *(Cubic irrational)*	287.8830223 *(Cubic irrational)*	-288 *(Rat.-based)*
$2-\sqrt{3}$ *(Rat.-based)*	$a^2+3.224586908a-0.935822227=0$	0.071796769 *(Rat.-based)*	0.864025456 *(Cubic irrational)*	-0.935822227 *(Cubic irrational)*

The angle ϕ is *trigonometrically* associated to α as follows:

Via the *Law of Sines* with respect to ΔAEF renders:

$$\frac{\sin(3\alpha+\phi)}{\overline{AF}}=\frac{\sin(\alpha-\phi)}{\overline{EF}}$$

$$\frac{\sin(3\alpha+\phi)}{3r}=\frac{\sin(\alpha-\phi)}{r}$$

Then,

$$\sin(3\alpha+\phi) = 3\sin(\alpha-\phi)$$
$$\sin(3\alpha)\cos\phi + \cos(3\alpha)\sin\phi = 3(\sin\alpha\cos\phi - \cos\alpha\sin\phi)$$
$$3\cos\alpha\sin\phi + \cos(3\alpha)\sin\phi = 3\sin\alpha\cos\phi - \sin(3\alpha)\cos\phi$$
$$[3\cos\alpha + \cos(3\alpha)]\sin\phi = [3\sin\alpha - \sin(3\alpha)]\cos\phi$$
$$\frac{\sin\phi}{\cos\phi} = \frac{3\sin\alpha - \sin(3\alpha)}{3\cos\alpha + \cos(3\alpha)}$$
$$\tan\phi = \frac{\sin^3\alpha}{\cos^3\alpha}$$
$$= \tan^3\alpha$$

Q.E.D.

As a *check*, given right triangle ABC whose hypotenuse is equal to 4r and whose $\angle BAC = 2\alpha$ *(Ref. Figure 43)*.

With *Point E* located at the middle of line $\overline{BC}$,

$$\tan\angle EAB = \frac{\overline{BE}}{\overline{AB}}$$
$$= \frac{1}{2}\tan(2\alpha)$$
$$= \frac{\tan\alpha}{1-\tan^2\alpha}\left(\frac{1+\tan^2\alpha}{1+\tan^2\alpha}\right)$$
$$= \frac{\tan\alpha + \tan^3\alpha}{1-\tan\alpha\tan^3\alpha}$$
$$= \frac{\tan\alpha + \tan\phi}{1-\tan\alpha\tan\phi}$$
$$= \tan(\alpha+\phi)$$
$$\angle EAB = (\alpha+\phi)$$

Q.E.D

The sides of the ***Angle Trisector Triangle*** may be *adjusted* in order to exhibit ***desired*** ratios with respect to one another. This is achieved simply by varying the included angle $3\alpha+\phi$.

Such *adjustment* enables *rationally-based and cubic irrational length combinations to coexist* within a *single geometry*. To demonstrate this, the following scenario is examined below:

With respect to *Figure 43*,

$\overline{FE} = r = \tan 20^o = \tan\theta$ *(Cubic irrational)*

$\overline{A'F} = 3r = 3\tan 20^o = 3\tan\theta$ *(Cubic irrational)*

$\overline{A'E} = a = \frac{\sqrt{3}}{2} = \sin 60^o$ *(Rationally-based)*

Substituting these values into *Complex Quadratic Equation 50* yields the following:

$$0 = a^2 - 2ar\cos(3\alpha + \phi) - 8r^2 \qquad \textit{[Ref. Equation 50]}$$

$$\cos(3\alpha + \phi) = \frac{a^2 - 8r^2}{2ar}$$

$$= \frac{(\frac{\sqrt{3}}{2})^2 - 8\tan^2\theta}{2(\frac{\sqrt{3}}{2})\tan\theta}$$

$$= \frac{3 - 32\tan^2\theta}{4\sqrt{3}\tan\theta}$$

$$= \text{-}0.491413881$$

$$3\alpha + \phi = 119.4335543^o$$

Respective values for α and φ are determined as follows:

$$\tan(3\alpha + \phi) = \tan[2\alpha + (\alpha + \phi)]$$

$$\tan 119.4335543^o = \frac{\tan(2\alpha) + \tan(\alpha + \phi)}{1 - \tan(2\alpha)\tan(\alpha + \phi)}$$

$$-1.77228647 = \frac{\tan(2\alpha) + \frac{1}{2}(2\alpha)}{1 - \frac{1}{2}\tan^2(2\alpha)}$$

$$= \frac{3\tan(2\alpha)}{2 - \tan^2(2\alpha)}$$

Then, by cross-multiplying:

$$(-1.77228647)[2 - \tan^2(2\alpha)] = 3\tan(2\alpha)$$

$$1.77228647\tan^2(2\alpha) - 3\tan(2\alpha) - 3.544572935 = 0$$

Via Quadratic Formula,

$$\tan(2\alpha) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+3 \pm \sqrt{9 - 4(1.77228647)(-3.544572939)}}{2(1.77228647)}$$

$$= 2.494494311$$

$$2\alpha = 68.15499704^o$$

$$\alpha = 34.07749852^o$$

$$3\alpha = 102.23224956^o$$

$$\tan\alpha = 0.676478302$$

$$\tan\phi = \tan^3\alpha$$
$$= (0.676478302)^3$$
$$= 0.309571958$$
$$\phi = 17.20105873^o$$
$$3\alpha + \phi = 102.22324956^o + 17.20105873^o$$
$$= 119.4335543^o$$ Q.E.D.

Checking the following *Complex Quadratic Equation* yields:

$$a^2 - 2ar\cos(3\alpha + \phi) - 8r^2 = 0$$ *[Ref. Equation 50]*

$$(\frac{\sqrt{3}}{2})^2 - 2(\frac{\sqrt{3}}{2})(\tan 20^o)\cos(3\alpha + \phi) - 8\tan^2 20^o = 0$$

Or, after multiplying through by -1/8,

$$\tan^2 20^o + (\frac{\sqrt{3}}{8})\cos(3\alpha + \phi)(\tan 20^o) - \frac{3}{32} = 0$$

$$\tan^2 20^o - (\frac{\sqrt{3}}{8})(0.491413881)\tan 20^o - \frac{3}{32} = 0$$

$$\tan^2 20^o - (0.106394226)\tan 20^o - \frac{3}{32} = 0$$

The above *Quadratic Equation* is of the form,

$$a'x^2 + bx + c = 0$$

Where *a′* is employed to differentiate from the term *a* above,

$$a' = 1$$
$$b = -0.106394226$$
$$c = -\frac{3}{32}$$

Hence tan 20° is determined via *Quadratic Formula* as follows:

$$\tan 20^o = \frac{1}{2a'}(-b \pm \sqrt{b^2 - 4a'c})$$
$$= \frac{1}{2(1)}[0.106394226 \pm \sqrt{(0.106394226)^2 - 4(1)(-3/32)}]$$
$$= 0.363970234; -0.257576009$$

Hence, the above analysis indicates that when a *Complex Quadratic,* ***Angle Trisector Triangle*** includes an angle of $3\alpha+\phi=119.4335543^o$ and a side $\overline{A'E}=a$ of *rationally-based length* $\sqrt{3}/2=\sin 60^o=\eta=\zeta/2$, its other two sides may be expressed by the following two cubic irrational lengths:

$$\overline{FE}=r=\tan 20^o=0.363970234$$
$$\overline{A'F}=3r=3\tan 20^o=1.091910703$$

In other words, a *cubic irrational number pair* may determined from another completely *independent* cubic irrational *number* in consonance with a *given rationally-based* number.

Since the above given *Complex Quadratic Equation* is *entirely Euclidean* in nature, a *sole compass and straight edge* may be used to *construct* tan 20° from the following given combination of *cubic irrational* and *rationally-based* values:

$$3\alpha+\phi=119.4335543^o$$
$$\overline{AD}=a=\frac{\sqrt{3}}{2}=\sin 60^o$$

This is accomplished in accordance with the *mapping process* specified in *Section 2.3* of this treatise. This is performed as follows:

For

$$a'=1$$
$$b=-0.1063942260 \qquad \text{(See above)}$$
$$c=-\frac{3}{32}$$

Where,

$$t=\sqrt{b^2-4a'c}$$
$$=\sqrt{(-0.1063942260)^2-4(1)(-\frac{3}{32})}$$
$$=0.621546242$$

From *Figure 44*,

$$\tan\rho=\frac{\overline{EF}}{t}=\frac{-b}{\overline{EF}}$$
$$\overline{EF}=\sqrt{-bt}$$
$$=0.257155461$$

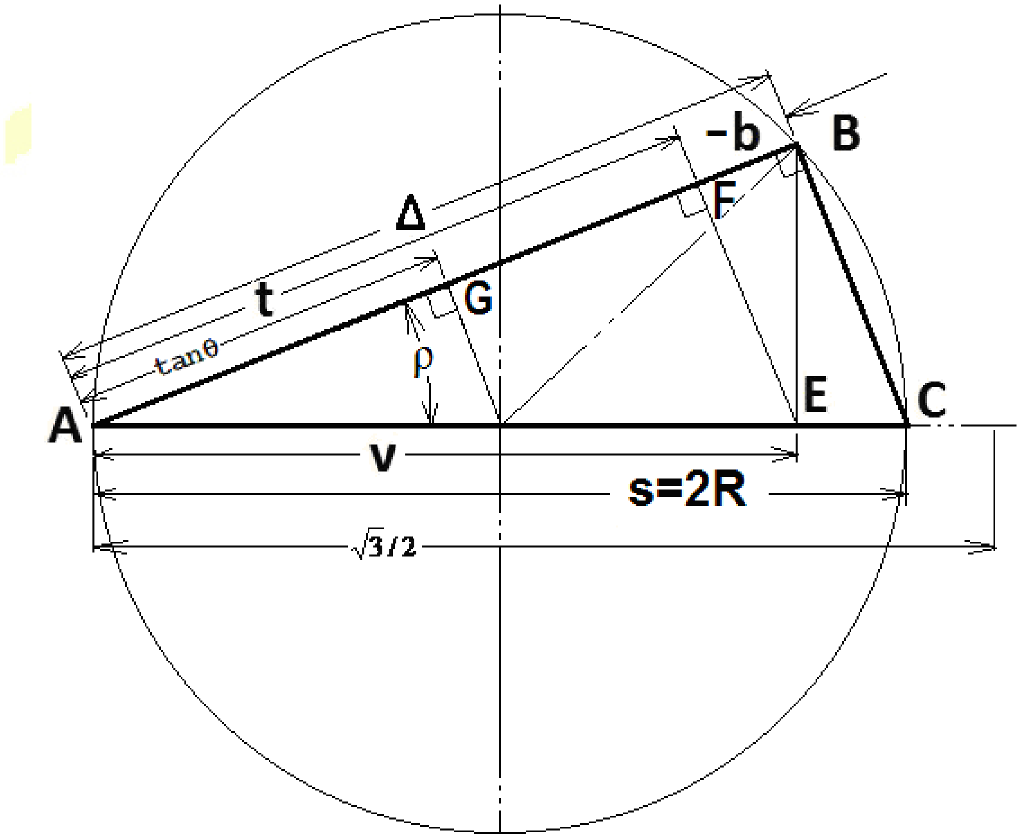
Therefore,

$$\tan\rho = \frac{\overline{EF}}{t} = \frac{0.257155461}{0.621546242}$$
$$= 0.413735042$$
$$\rho = 22.47659407^{o}$$

Then, Δ and r are calculated as follows:

$$\Delta = t - b$$
$$= 0.621546242 - (-0.1063942260)$$
$$= 0.727940468$$

Figure 44. Geometric Construction showing Relationship between ζ/2 and tanθ.



$$\cos\rho = \cos 22.47659404^{o} = \frac{\Delta}{s} = \frac{\Delta}{2R} = 0.924035785$$
$$\frac{\Delta}{2(0.924035785)} = R$$
$$\frac{0.727940468}{2(0.924035785)} = R$$
$$0.393891924 = R$$

Where,

$$\Delta = t - b$$
$$= \sqrt{b^2 - 4a'c} - b$$
$$= 2(\frac{1}{2})(-b + \sqrt{b^2 - 4a'c'})$$
$$= 2(-\frac{b}{2} + \sqrt{b^2 - 4a'c})$$
$$= 2\tan 20^o$$

Thereafter designating tan 20° as tan θ renders,

$$\Delta = 2\tan\theta$$

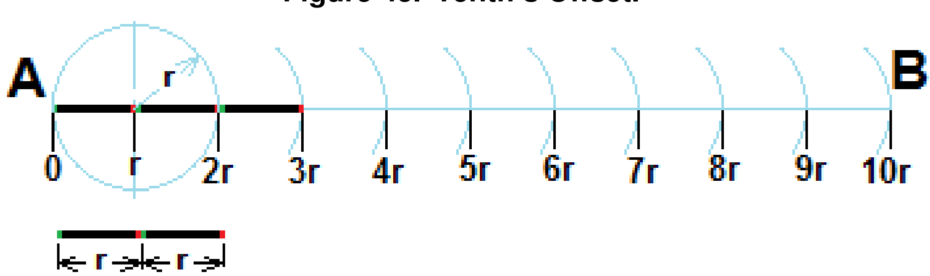
Figure 44 portrays tan θ equal to one-half the length of Δ. Construction is accomplished by first using a compass to draw a circle whose radius "R" is equal to the cubic irrational value of 0.393891924 units. This is easily accomplished by realizing that a *ruler* may be developed solely via *Euclidean process* as presented below. Then with respect to *Figure 44*, a length Δ which is exactly 0.727940468 units long is marked off onto the circumference of the aforementioned circle.

Hence, right triangle ABC which is inscribed within the circle now can be completed by connecting line $\overline{BC}$. Thereafter, a length $\sqrt{3}/2$ which equals 1.732050808/2, or 0.866025403 units is marked off to show the linear relationship between $\sqrt{3}/2$ and $\tan\theta$.

Figure 45 illustrates the principle where any arbitrary length, notated as r in this case, is 'set off', or multiplied by itself ten times to form straight line segment $\overline{AB}$. Then,

- Designated length r is equal to one-tenth (1/10) of the marked off length
- The entire offset is of length is equal to 10r

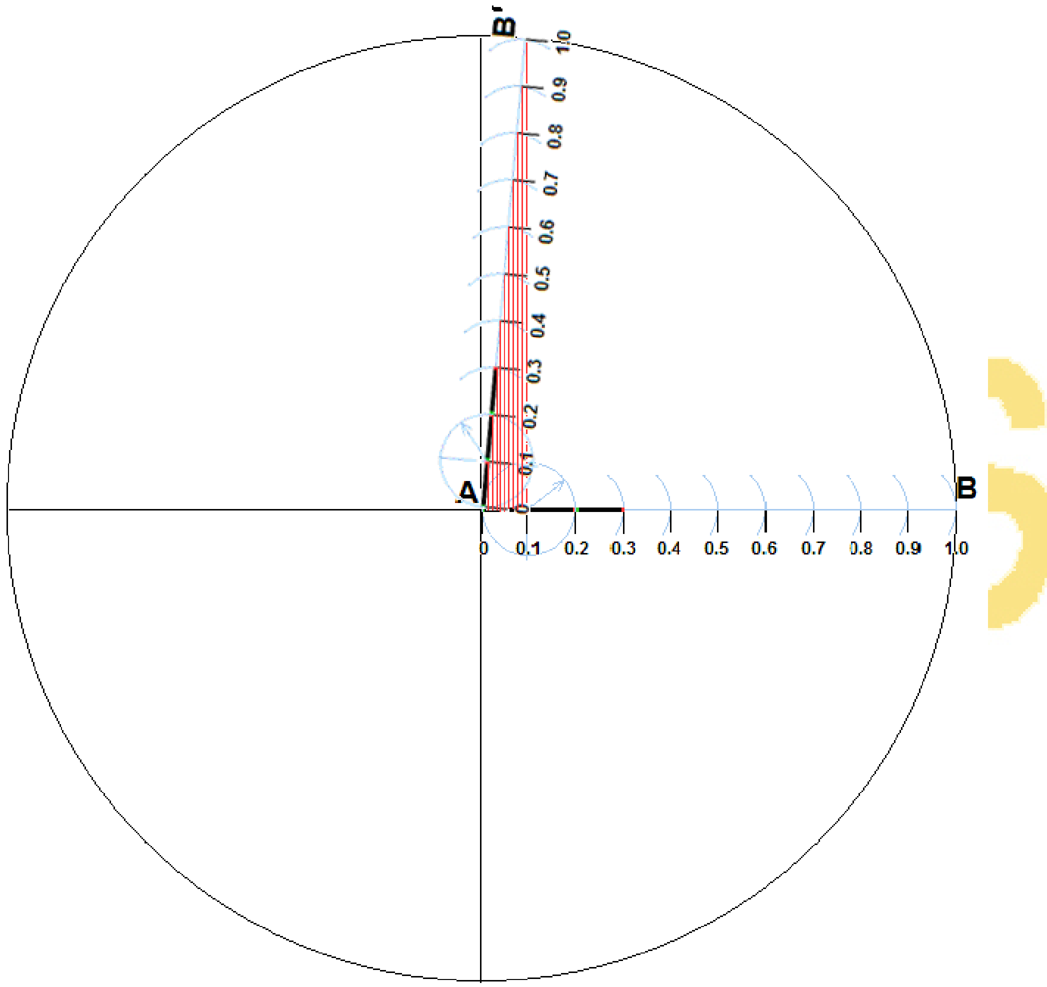
Figure 45. Tenth's Offset.



As the length of 10r is arbitrarily set to one unit in length, each increment r represents one-tenth of that value, or simply 1/10. As such, *Figure 45* then depicts, or characterizes a *ruler*.

Further *ruler* division into hundredth's of a unit also is easily accomplished as shown in *Figure 46*. Therein, the entire contents of *Figure 45* are rotated about the origin of a circle located at point A of radius $\overline{AB}$ until point B aligns with the r = 0.1 unit vertical projection on the circumference of the circle, designated as point B'.

Figure 46. Hundredth's Offset.



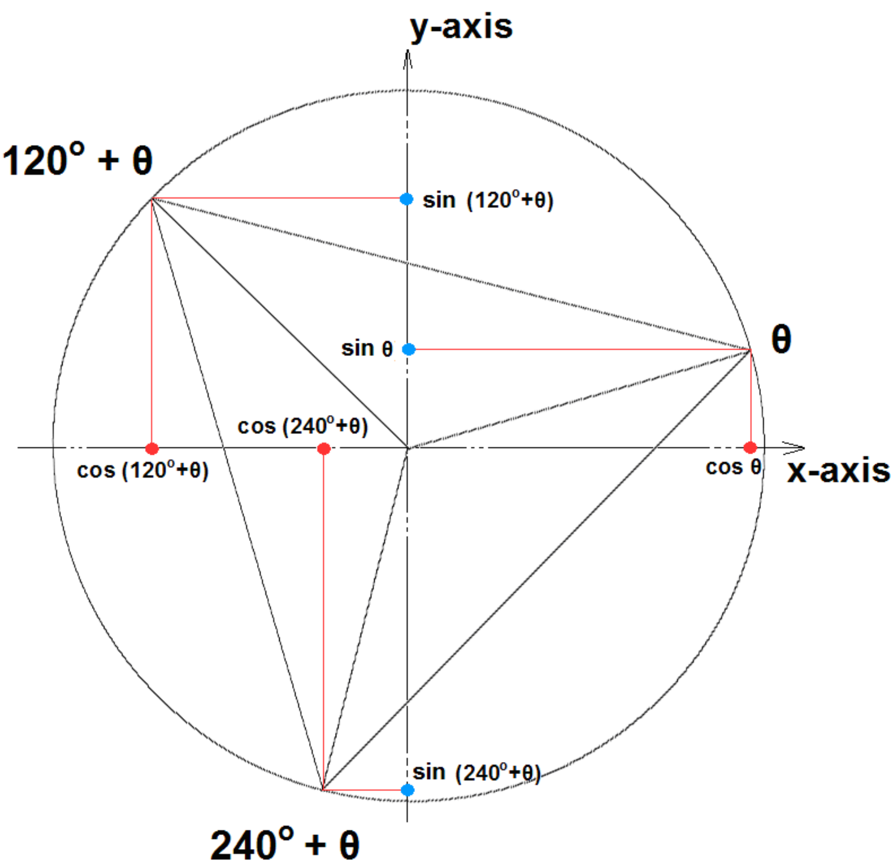
In conclusion, it is contended that *cubic irrational numbers, or cubic irrational lengths*, appear as *pairs or conjugates* in *Complex Quadratic Equations* where one may be determined via the other.

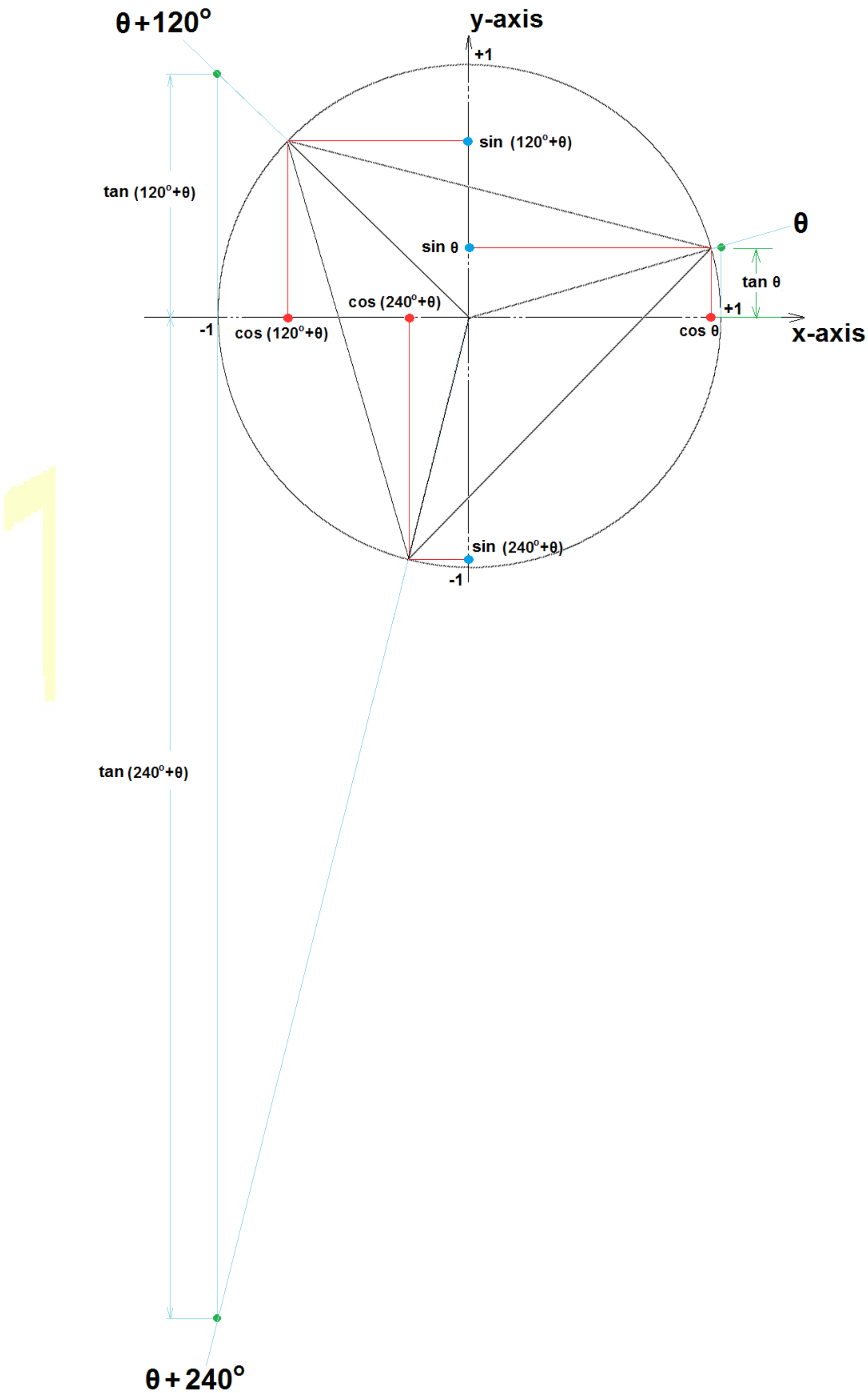
18.6. Equations Emulated by the Cosine Circle.

As portrayed in *Figure 47,* the ***Cosine Circle*** consists of an *equilateral triangle* whose vertices lie upon the circumference of a *unit circle*.

Such construction is achieved simply by locating the *origin of the unit circle* at the intersection point of the respective bisectors of the three angles resident within an *equilateral triangle*. A radius then is selected such that the circumference of the *unit circle* passes through all vertices of the *equilateral triangle*.

Figure 47. Cosine Circle Construction.




θ+120°
y-axis
+1
sin (120°+θ)
tan (120°+θ)
θ
sin θ
tan θ
cos (240°+θ)
+1
-1
cos (120°+θ)
cos θ
x-axis
sin (240°+θ)
-1
tan (240°+θ)
θ+240°

As indicated:

- The *three roots* for the *Known Cubic Equation for the Cosine 3θ* (Ref.*Equation 1*) designate respective *x-axis values* where vertical projections constructed from the three vertices of the inscribed equilateral triangle intersect the abscissa *(Ref. Upper Figure)*
- The *three roots* for the *Known Cubic Equation for the Sine 3θ* (Ref. *Equation 2*) designate respective *y-axis values* where horizontal projections constructed from the three vertices of the inscribed equilateral triangle intersect the ordinate *(Ref. Upper Figure)*
- The *three roots* for the *Known Cubic Equation for the Tangent 3θ* (Ref. *Equation 3*) designate respective ordinate values where unit circle radii passing through the three vertices of the inscribed equilateral triangle intersect vertical projections which pass through the points $x = \pm 1$ *(Ref. Lower Figure)*

As such, the *Cosine Circle* emulates the following three equations:

Upper Figure:

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta) \qquad \textit{[Ref. Equation 1]}$$

With roots *(Ref. Section 2.4.1)*:

$$x_1 = \cos\theta$$

$$x_2 = \cos(\theta + 120^o)$$

$$x_3 = \cos(\theta + 240^o)$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta) \qquad \textit{[Ref. Equation 2]}$$

With roots *(Ref. Section 2.4.2)*:

$$y_1 = \sin\theta$$

$$y_2 = \sin(\theta + 120^o)$$

$$y_3 = \sin(\theta + 240^o)$$

Lower Figure:

$$\tan^3\theta = 3\tan\theta - \tan(3\theta)(1 - 3\tan^2\theta) \qquad \textit{[Ref. Equation 3]}$$

With roots *(Ref. Section 2.4.3)*:

$$z_1 = \tan\theta$$

$$z_2 = \tan(\theta + 120^o)$$

$$z_3 = \tan(\theta + 240^o)$$

In each of the *three equations* presented above, *coefficients* appear either as integers or multiplies of *trigonometric values* that apply to the angle 3θ.

All such *coefficients* are determinable from *respective root sets* which characterize each of the *nine supporting equations* listed below. Hence such *additional equations* also relate to the *Figure 47 cosine circle geometry.*

- The following x_1, x_2, x_3 *root set* values may be applied via *Euclidean construction* to determine respective *right-hand term values* listed in each of the three equations shown below:

$$x_1 x_2 x_3 = \frac{\cos(3\theta)}{4} \qquad \text{[Ref. Equation 5]}$$

$$x_1 + x_2 + x_3 = 0 \qquad \text{[Ref. Equation 6]}$$

$$x_1 x_2 + x_1 x_3 + x_2 x_3 = -\frac{3}{4} \qquad \text{[Ref. Equation 7]}$$

- The following y_1, y_2, y_3 *root set* values may be applied via *Euclidean construction* to determine respective *right-hand term values* listed in each of the three equations shown below:

$$y_1 y_2 y_3 = -\frac{\sin(3\theta)}{4} \qquad \text{[Ref. Equation 8]}$$

$$y_1 + y_2 + y_3 = 0 \qquad \text{[Ref. Equation 9]}$$

$$y_1 y_2 + y_1 y_3 + y_2 y_3 = -3/4 \qquad \text{[Ref. Equation 10]}$$

- The following z_1, z_2, z_3 *root set* values may be applied via *Euclidean construction* to determine respective *right-hand term values* listed in each of the three equations shown below:

$$z_1 z_2 z_3 = -\tan(3\theta) \qquad \text{[Ref. Equation 11]}$$

$$z_1 + z_2 + z_3 = 3\zeta \qquad \text{[Ref. Equation 12]}$$

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = -3 \qquad \text{[Ref. Equation 13]}$$

In *Figure 47*, as the inscribed *equilateral triangle* is rotated about the unit circle, the angle θ varies in magnitude, but always incorporates the x-axis as its lower boundary.

Cosine Circle geometric resolution consists of rotating the inscribed equilateral triangle until one of its vertices coincides or aligns with a determined angle of either $\theta + 120^o$, or $\theta + 240^o$. The angle θ then becomes represented by the angle described between the x-axis and a line drawn from the origin to the equilateral triangle's free vertex *(Ref. Figure 47)*.

The *cosine circle* may be represented by equations of any *category*; where, an example for each is presented below:

A *Category 1 equation type* results when $3\theta = 45^o$ such that:

$$\cos(2\theta) = \cos 30^o = 2\cos^2\theta - 1 = \frac{\sqrt{3}}{2}$$

$$\cos\theta = \cos 15^o = \sqrt{\frac{\frac{\sqrt{3}}{2} + 1(\frac{2}{2})}{2}} = \frac{\sqrt{\sqrt{3}+2}}{2}$$

$$\sin(2\theta) = \sin 30^o = 2\sin\theta\cos = \frac{1}{2}$$

$$\sin\theta = \sin 15^o = \frac{1}{4\cos\theta} = \frac{1}{2\sqrt{\sqrt{3}+2}}$$

$$\cos(\theta + 120^o) = \cos\theta(-1/2) - \sin\theta(\sqrt{3}/2) = -\frac{\sqrt{\sqrt{3}+2}}{4} - \frac{\sqrt{3}}{4\sqrt{\sqrt{3}+2}} = -\frac{\sqrt{3}+1}{2\sqrt{\sqrt{3}+2}}$$

$$\cos(\theta + 240^o) = \cos\theta(-1/2) + \sin\theta(\sqrt{3}/2) = -\frac{\sqrt{\sqrt{3}+2}}{4} + \frac{\sqrt{3}}{4\sqrt{\sqrt{3}+2}} = -\frac{1}{2\sqrt{\sqrt{3}+2}}$$

$$x_1 + x_2 + x_3 = 0$$ *[Ref. Equation 6]*

$$cos\,\theta + cos(\theta + 120^o) + cos(\theta + 240^o) = 0$$

$$\frac{\sqrt{\sqrt{3}+2}}{2} - \frac{\sqrt{3}+1}{2\sqrt{\sqrt{3}+2}} - \frac{1}{2\sqrt{\sqrt{3}+2}} = 0$$

$$\frac{(\sqrt{3}+2) - (\sqrt{3}+1) - 1}{2\sqrt{\sqrt{3}+2}} = 0$$

$$0 = 0$$

As indicated, all terms indicated in the above result are *rationally-based* numbers. Moreover, *Equation 6* qualifies as a *Category 1 Equation type* because all of its *coefficients* also can be considered to equal unity.

A *Category 2 equation type* results when $3\theta = 37^o$ such that:

$$\sin(3\theta) = 0.601815023$$

$$y_1 = \sin\theta = 0.213598772$$

$$y_2 = \sin(\theta + 120^o) = 0.739239427$$

$$y_3 = \sin(\theta + 240^o) = -0.952838199$$

When the *coefficient* of the *left-hand term* is considered to be equal to the value of *any* of the roots, and the *coefficient* of the *right-hand term* is considered to be equal to the value of the *entire term*:

$$y_1 y_2 y_3 = -\frac{\sin(3\theta)}{4}$$ *[Ref. Equation 8]*

$$0.213598772(y_2 y_3) = -0.150453755$$

A *Category 3 equation type* results when $3\theta = 37^o$ such that the *coefficient* of the *left-hand term* is considered to be equal to the value of *any* of the roots, and the *coefficient* of the *right-hand term* is considered to be equal to 1/4:

$$y_1 y_2 y_3 = -\frac{\sin(3\theta)}{4}$$ *[Ref. Equation 8]*

$$0.739239427(y_1 y_3) = -\frac{1}{4}\sin(3\theta)$$

SECTION 19. *RST EXPLANATION FOR EUCLIDEAN TRISECTION PROBLEM.*

Where the *3θ Cubic Equation* is determined as a mere reformatting of *Equation 3*

$$\zeta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad \text{[Ref. Equation 3]}$$

$$\zeta(1 - 3\tan^2\theta) = 3\tan\theta - \tan^3\theta$$

$$\tan^3\theta - 3\zeta\tan^2\theta - 3\tan\theta + \zeta = 0$$

$$z^3 - (3\zeta)z^2 - 3z + \zeta = 0 \qquad \text{[3θ Cubic Equation]}$$

It then must exhibit its same *root structure (Ref. Section 2.4.3)*:

- z_1 = tan θ
- z_2 = tan (θ + 120°)
- z_3 = tan (θ + 240°)

Being that the *3θ Cubic Equation* represents a particular, very definable form of the *Generalized Cubic Equation (GCE)* in itself, it may be referred to as a ***primary GCE***.

Moreover when $R = 1$, many other ***secondary, independent GCE's*** exist, each of which may be depicted as possessing the *root structure*:

$$z_R = R\tan\theta = \tan\theta_R = (1)\tan\theta = \tan\theta$$

$$z_S = S\tan\theta = \tan\theta_S$$

$$z_T = T\tan\theta = \tan\theta_T$$

Such that,

$$\theta_R + \theta_S + \theta_T = 3\theta$$

$$\theta + \theta_S + \theta_T = 3\theta$$

$$\theta_S + \theta_T = 2\theta$$

Since both of these formats share a ***common root*** $z_1 = z_R = \tan\theta$, once they become *properly associated* or *paired off,* with all their *coefficients* being *fully characterized*, then they can be mathematically reduced into ***quadratic form*** and thereby ***simultaneously resolved*** via the *Euclidean mapping process* stipulated in *Section 2.3*.

This entire method consists of:

1) *Identifying* an *angle 3θ* for analysis;
2) *Geometrically constructing* its *tangent* $\zeta = \tan(3\theta)$;
3) *Specifying* the *3θ Cubic Equation*; and
4) *Specifying* an *associated second, independent GCE*.

Respective coefficients of such *second, independent* GCE can be determined in accordance with *Equation 51* as follows:

Equation 51. ***Coefficient Structure* of a *Second, Independent GCE for R=1.***

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma}$$

1) $z_R = \tan\theta$ is calculated *trigonometrically* from $\zeta = \tan(3\theta)$
2) A *designated value* of β becomes arbitrarily assigned
3) *Coefficient* γ then is readily calculated
4) *Remaining coefficient* δ is calculated via *Equation 36* as follows*:*

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \quad \textit{[Ref. Equation 36]}$$

$$\zeta(1 - \gamma) + \beta = \delta$$

Making use of the relationship $\theta_S + \theta_T = 2\theta$ determined above, *Equation 51* is derived as follows:

Where the *3θ Cubic Equation format* is reiterated below:

$$z_R^{\;3} - 3\zeta z_R^{\;2} - 3z_R + \zeta = 0$$

$$z_R^{\;3} = 3\zeta z_R^{\;2} + 3z_R - \zeta$$

For the *Generalized Cubic Equation* when $\alpha = 1$,

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$z_R^{\;3} + \beta z_R^{\;2} + \gamma z_R + \delta = 0$$

Since both above equations share a ***common root*** $\boldsymbol{z_R}$**:**

Substitution of the above *3θ Cubic Equation* result gives:

$$(3\zeta z_R^{\;2} + 3z_R - \zeta) + \beta z_R^{\;2} + \gamma z_R + \delta = 0$$

$$(3\zeta + \beta)z_R^{\;2} + (3 + \gamma)z_R + (\delta - \zeta) = 0$$

Since,

$$\theta_S + \theta_T = 2\theta$$

$$\tan(\theta_S + \theta_T) = \tan(2\theta)$$

$$\frac{\tan\theta_S + \tan\theta_T}{1 - \tan\theta_S \tan\theta_T} = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\frac{z_S + z_T}{1 - z_S z_T} = \frac{2z_R}{1 - z_R^{\;2}}$$

Cross multiplication renders,

$$(z_S + z_T)(1 - z_R^{\;2}) = 2z_R(1 - z_S z_T)$$

$$(z_S + z_T)(1 - z_R^{\;2}) = 2z_R + 2\delta$$

$$-2\delta = 2z_R + (z_S + z_T)(z_R{}^2 - 1)$$
$$= 2z_R - (\beta + z_R)(z_R{}^2 - 1)$$
$$= (3z_R - z_R{}^3) - \beta(z_R{}^2 - 1)$$

Such that,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \quad \textit{[Ref. Equation 36]}$$

$$\zeta(1-\gamma) = \delta - \beta$$
$$-\delta = -\beta - \zeta(1-\gamma)$$
$$-2\delta = -2\beta - 2\zeta(1-\gamma)$$

Via substitution,

$$-2\beta - 2\zeta(1-\gamma) = (3z_R - z_R{}^3) - \beta(z_R{}^2 - 1)$$
$$2\zeta(1-\gamma) + 2\beta = -(3z_R - z_R{}^3) + \beta(z_R{}^2 - 1)$$
$$2\zeta - 2\zeta\gamma + 2\beta = -\zeta(1 - 3z_R{}^2) + \beta(z_R{}^2 - 1)$$
$$3\zeta - 2\zeta\gamma + 3\beta = (3\zeta + \beta)z_R{}^2$$
$$3(\zeta + \beta) - 2\zeta\gamma = (3\zeta + \beta)z_R{}^2$$

The above *right-hand term* has exactly the same value as the first term of the left-hand member listed in *Generalized Cubic Equation* reduction shown above and restated below:

$$(3\zeta + \beta)z^2 + (3+\gamma)z + (\delta - \zeta) = 0$$

Substitution renders:

$$3(\zeta + \beta) - 2\zeta\gamma + (3+\gamma)z_R + (\delta - \zeta) = 0$$
$$(3+\gamma)z_R = 2\zeta\gamma - 3(\zeta + \beta) - (\delta - \zeta)$$
$$z_R = \frac{2\zeta\gamma - 3(\zeta + \beta) - (\beta - \zeta\gamma)}{3+\gamma}$$

Or,

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3+\gamma} \qquad \textit{Q.E.D.}$$

Above a <u>singular value</u> for the *unknown coefficient* γ is readily obtained by first ascribing ***properly associated*** $\zeta = \tan(3\theta)$ and $z_R = \tan\theta$ *trigonometric values*, and thereafter assigning an arbitrary value to β.

However, when <u>not</u> relying upon the fact that $z_R = \tan\theta$ can be *trigonometrically* determined from $\zeta = \tan(3\theta)$, such *common root value* instead must be ascertained from the two remaining *unknown values* β and γ *(Ref. Equation 51)*.

This can be accomplished only when such ***correct*** <u>singular value</u> of γ becomes interposed into *Equation 51* with respect to each and every specific value of $\zeta = \tan(3\theta)$ and arbitrarily assigned value of β which also become applied to it.

Unfortunately, in most cases, without having ***advance knowledge*** of such $z_R = \tan\theta$ to $\zeta = \tan(3\theta)$ *trigonometric relationship*, it becomes impossible to distinguish the proper value of γ that should become inserted in the first place.

In other words, γ then would become distinguishable by *Equation 51* only after properly associated values of ζ and the ***unknown common root*** z_R, along with an arbitrarily assigned value of β *first* become disclosed.

- ***More specifically restated***:
  Aforehand knowledge of such *common root value* z_R, would be needed in order to enable determination of the respective values of ***coefficients*** which belong to, or fully characterize such coterie of *second, independent GCE's*
- ***Even more fully explained***:
  A *second, independent GCE*, considered to be a *Cubic Equation* whose coefficients could be fed into the linear *Equation 51* for purposes of obtaining a ***common root value*** z_R that, in turn, could be operated upon via *geometric construction* in order to produce a *trisected angle θ*, cannot be determined without having ***aforehand knowledge*** of such ***common root value*** z_R in the first place

Such *preponderance* poses an *insurmountable difficulty* or *unfathomable discontinuity* for the ***Euclidean process*** which ***must*** be told exactly which *coefficient values* are to applied in the first place when attempting to *geometrically construct* a ***common root*** z_R.

Therefore, ***it is concluded*** that when a *coefficient structure* for a *second, independent GCE*:

1) Can be determined without gaining ***aforehand knowledge*** of the value of its ***common root*** z_R, then such equation *can* be used to *reduce* its *associated 3θ Cubic Equation* into *quadratic form*, thereby enabling a ***simultaneous resolution*** via the *geometric mapping process specified in Section 2.3*; which in turn enables the depiction of an *angle θ* which represents a bonafide *trisector* for any *given, or assigned angle 3θ (Ref. Section 20)*; or

2) Cannot be determined without gaining ***aforehand knowledge*** of the value of its ***common root*** z_R, then such equation *cannot* be fed into *linear Equation 51* for purposes of obtaining a *common root value* z_R that, in turn, could have been operated upon via *geometric construction* in order to produce a *trisected angle θ*.

An example which further explains the impact of this second above *precept* is afforded below for the rare circumstance when the *3θ Cubic Equation* contains a *rational root* while exhibiting only *rational coefficients*:

$$z_R^{\ 3} - (\frac{13}{3})z_R^{\ 2} - 3z_R + \frac{13}{9} = 0$$

The *format* of such *3θ Cubic Equation* is verified below:

$$z_R^{\ 3} - 3(\frac{13}{9})z_R^{\ 2} - 3z_R + \frac{13}{9} = 0$$

$$z^3 - (3\zeta)z^2 - 3z + \zeta = 0 \qquad \textit{[3θ Cubic Equation]}$$

$$\zeta = \frac{3z_R - z_R^{\ 3}}{1 - 3z_R^{\ 2}}$$

And since $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$:

$$\zeta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \qquad \textit{[Ref. Equation 3]}$$

Its *rational* ***common root*** is trigonometrically determined below:

$$\zeta = \tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

$$\theta = 18.43494882^o$$

$$z_R = \tan\theta = 1/3$$

As such, it is obvious that:

- The *coefficients* contained in the *3θ Cubic Equation* presented above, in addition to the value of $\zeta = \tan(3\theta) = 13/9$, all represent *rational values* and, hence, can be ***geometrically constructed*** as lengths by means of applying a *straightedge* and *compass* alone.

  This is because they all *stem* from any given or assigned length of unity *(Ref. Section 9.1)*.
- The ***common root value*** $z_R = \tan\theta = 1/3$ also is a *rational length;* whereby, portrayal of the *trisected angle θ*, in this particular case, also rather easily could be produced via ***geometric construction*** using only *Euclidean tools*.

However, <u>no</u> ***geometric construction method*** exists which can determine $z_R = 1/3$ when <u>only</u> a known *value of* $\zeta = \tan(3\theta) = 13/9$ is supplied or *given* in the first place.

By introduction of *Equation 51*, such above stated impossibility is explained *mathematically* as follows:

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma} \qquad \textit{[Ref. Equation 51]}$$

$$\frac{1}{3} = \frac{3(\frac{13}{9})(\gamma - 1) - 4\beta(\frac{3}{3})}{3 + \gamma}$$

$$3 + \gamma = 13(\gamma - 1) - 12\beta$$

For the specific case when $\beta = 0$:

$$3 + \gamma = 13(\gamma - 1) - 12\beta$$

$$3 + \gamma = 13(\gamma - 1) - 12(0)$$

$$3 + \gamma = 13\gamma - 13$$

$$16 = 12\gamma$$

$$\frac{4}{3} = \gamma$$

$$\delta = \zeta(1 - \gamma) + \beta$$

$$= \frac{13}{9}[1(\frac{3}{3}) - \frac{4}{3}] + 0$$

$$= -\frac{13}{9}(\frac{1}{3})$$

$$= -\frac{13}{27}$$

As such, one real *second, independent GCE* for $R = \alpha = 1$ is:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + \frac{4}{3}z - \frac{13}{27} = 0$$

However, the above determination could not have been rendered without first having received ***aforehand knowledge*** of the *common rational* **common root value** $z_R = 1/3$.

Quite obviously, this approach is <u>not</u> permitted when attempting to *trisect* an angle via *Euclidean means*!

Such *second independent GCE* for $\alpha = 1$, as determined above, is easily verified by resolving a *quadratic reduction* which is formed by merging *primary* and *secondary GCE's* using the formula reiterated below *(Ref. Equation 51 derivation)*:

$$(3\zeta + \beta)z_R^{\ 2} + (3+\gamma)z_R + (\delta - \zeta) = 0$$

$$[3(\frac{13}{9}) + 0]z_R^{\ 2} + [3(\frac{3}{3}) + \frac{4}{3}]z_R - [\frac{13}{27} + (\frac{13}{9})(\frac{3}{3})] = 0$$

$$\frac{13}{3}z_R^{\ 2} + \frac{13}{3}z_R - \frac{52}{27} = 0$$

$$z_R^{\ 2} + z_R - \frac{4}{9} = 0$$

$$z_R^{\ 2} + z_R + (\frac{1}{2})^2 = \frac{1}{4}(\frac{9}{9}) + \frac{4}{9}(\frac{4}{4})$$

$$(z_R + \frac{1}{2})^2 = \frac{9+16}{36}$$

$$z_R = \frac{1}{6}(-3 \pm 5)$$

$$= \frac{1}{3}; -\frac{4}{3}$$

This above analysis does not consider the possibility of obtaining a GCE set of *identifiable* ***common root*** *equations* when $R \neq 1$.

Equation 51 has many practical applications. For instance:

- It validates the *3θ Cubic Equation* by substituting the value of its *third term coefficient* $\gamma = -3$ into *Equation 51* as follows:

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$z_R(3+\gamma) = 3\zeta(\gamma-1) - 4\beta$$
$$z_R(3-3) = 3\zeta(-3-1) - 4\beta$$
$$0 = 3\zeta(-4) - 4\beta$$
$$4\beta = 3\zeta(-4)$$
$$\beta = -3\zeta$$

- It validates that *Generalized Cubic Equations* whose ***sub-element*** $R = 1$ contain a root whose value is equal to the negative of its *β coefficient* when $\gamma = +1$ as follows:

$$z_R(3+\gamma) = 3\zeta(\gamma-1) - 4\beta$$
$$z_R(3+1) = 3\zeta(1-1) - 4\beta$$
$$z_R = -\beta$$

Another *second independent GCE* for $R=1$ example is featured below to *emphasize that knowledge* of the ***common root value*** z_R is needed ***aforehand*** in order to characterize its respective *coefficient structure*. Given that:

$$\theta_R = \theta$$
$$\theta_S = \theta + 45^o$$
$$\theta_T = \theta - 45^o$$
$$\Sigma = \theta_R + \theta_S + \theta_T = 3\theta$$

Such that,

$$z_R = \tan\theta_R = \tan\theta$$

$$z_S = \tan\theta_S = \tan(\theta + 45^o) = \frac{\tan\theta + 1}{1 - \tan\theta}$$

$$z_T = \tan\theta_T = \tan(\theta - 45^o) = \frac{\tan\theta - 1}{1 + \tan\theta}$$

$$\begin{aligned} z_S + z_T &= \frac{\tan\theta + 1}{1 - \tan\theta} + \frac{\tan\theta - 1}{1 + \tan\theta} \\ &= \frac{(\tan\theta + 1)^2 - (\tan\theta - 1)^2}{1 - \tan^2\theta} \\ &= \frac{(\tan^2\theta + 2\tan\theta + 1) - (\tan^2\theta - 2\tan\theta + 1)}{1 - \tan^2\theta} \\ &= \frac{4\tan\theta}{1 - \tan^2\theta} \end{aligned}$$

$$\begin{aligned} z_S z_T &= (\frac{\tan\theta + 1}{1 - \tan\theta})(\frac{\tan\theta - 1}{1 + \tan\theta}) \\ &= \frac{\tan^2\theta - 1}{1 - \tan^2\theta} \\ &= \frac{-(1 - \tan^2\theta)}{1 - \tan^2\theta} \\ &= -1 \end{aligned}$$

Since,

$$\begin{aligned} \beta &= -(z_R + z_S + z_T) \\ &= -(\tan\theta + \frac{4\tan\theta}{1 - \tan^2\theta}) \\ &= -(\frac{5\tan\theta - \tan^3\theta}{1 - \tan^2\theta}) \end{aligned}$$

$$\begin{aligned} \gamma &= z_R(z_S + z_T) + z_S z_T \\ &= \tan\theta(\frac{4\tan\theta}{1 - \tan^2\theta}) - 1 \end{aligned}$$

$$\begin{aligned} \delta &= -z_R z_S z_T \\ &= -\tan\theta(-1) \\ &= \tan\theta \end{aligned}$$

The *second independent GCE* for $\alpha = 1$ is:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \text{[Ref. Equation 32]}$$

$$z_R{}^3 + \beta z_R{}^2 + \gamma z_R + \delta = 0$$

$$z_R{}^3 - (\frac{5\tan\theta - \tan^3\theta}{1-\tan^2\theta}) z_R{}^2 + [\tan\theta(\frac{4\tan\theta}{1-\tan^2\theta}) - 1] z_R + \tan\theta = 0$$

Clearly, all the *coefficients* enumerated above are represented as functions of the root, $z_R = \tan\theta$. Hence, the *coefficient values* of such *Generalized Cubic Equation* cannot be determined without having aforehand knowledge of its ***common root value*** z_R.

Conversely, even when *three roots* and *all coefficients* belonging to a *second, independent GCE* for $R = 1$ can be ***geometrically constructed***, *trisecting* an associated *given 3θ angle* still remains intractable. To emphasize this point, consider the specific condition when:

$$z_R = R\tan\theta = \tan\theta_R = -\frac{1}{\sqrt{3}}$$

$$z_S = S\tan\theta = \tan\theta_S = 1$$

$$z_T = T\tan\theta = \tan\theta_T = 1$$

Then,

$$\theta_R = -30^o$$

$$\theta_S = 45^o$$

$$\theta_T = 45^o$$

$$\Sigma = 3\theta = 60^o$$

$$\zeta = \tan(3\theta) = \sqrt{3}$$

$$\theta = \frac{60^o}{3} = 20^o$$

$$\tan\theta = 0.363970234$$

For $\alpha = 1$:

$$\beta = -(z_R + z_S + z_T) = -(-\frac{1}{\sqrt{3}} + 1 + 1) = -1.422649731$$

$$\gamma = z_R(z_S + z_T) + z_S z_T = -\frac{1}{\sqrt{3}}(1+1) + (1)(1) = -0.154700538$$

$$\delta = -z_R z_S z_T = -(-\frac{1}{\sqrt{3}})(1)(1) = 0.577350269$$

Then,

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 - 1.422649731z^2 - 0.154700538z + 0.577350269 = 0$$

Check,

$$z_R{}^3 - 1.422649731z_R{}^2 - 0.154700538z_R + 0.577350269 = 0$$

$$(-\frac{1}{\sqrt{3}})^3 - 1.422649731(-\frac{1}{\sqrt{3}})^2 - 0.154700538(-\frac{1}{\sqrt{3}}) + 0.577350269 = 0$$

$$-0.192450089 - 0.474216577 + 0.089316397 + 0.577350269 = 0$$

$$-\frac{2}{3} + \frac{2}{3} = 0$$

$$0 = 0$$

$$(z_S; z_T)^3 - 1.422649731(z_S; z_T)^2 - 0.154700538(z_S; z_T) + 0.577350269 = 0$$

$$(1)^3 - 1.422649731(1)^2 - 0.154700538(1) + 0.577350269 = 0$$

$$1 - 1.422649731 - 0.154700538 + 0.577350269 = 0$$

$$-0.422649731 + 0.422649731 = 0$$

$$0 = 0$$

All above determined *coefficients* <u>can</u> be *constructed* via *Euclidean means* since they all are *rationally-based (Ref. Section 9.1)*; that is, they represent *mathematical combinations* of the *associated GCE root structure* consisting of the *rationally-based* values 1, 1, and $-1/\sqrt{3}$.

However, such *associated GCE* contains ***no*** roots *in common* with its respective *3θ Cubic Equation*. Hence, $R \neq 1$ and such resulting equation <u>cannot</u> qualify as a *second, independent GCE* for $R = 1$. This is evidenced by the *root structure* for each as presented below:

3θ Cubic Equation Roots	*Associated GCE Roots*
$z_1 = \tan\theta_R = \tan\theta = 0.363970234$	$z_R = R\tan\theta = -1/\sqrt{3}$
$z_2 = \tan\theta_S = \tan(\theta + 120^o) = -0.839099631$	$z_S = S\tan\theta = 1$
$z_3 = \tan\theta_T = \tan(\theta + 240^o) = 5.67128182$	$z_T = T\tan\theta = 1$

Above, ***associated GCE*** *roots* are represented as *Complex Linear Equations* expressing $\tan\theta$ and respective values of *R*, *S*, and *T*; all unknown terms that <u>cannot</u> be deciphered by *Euclidean* means.

This means that *many* values of *R*, for example, can be arbitrarily introduced, such that *compensating values* of $\tan\theta$ must equal $-1/(\sqrt{3}R)$. Moreover, only *one* unknown value of -1.586256828 for *R* correctly determines $\tan\theta = 0.363970234$.

The fact that a *given 3θ Cubic Equation* can be reduced by any *second, independent GCE* of $R=1$ which shares its ***common root*** $\boldsymbol{z_R}$ in order to ***simultaneously resolve*** is demonstrated as follows for the particular condition when $3\theta = 60^o$:

$$z_R{}^3 - 3\zeta z_R{}^2 - 3z_R + \zeta = 0$$

$$z_R{}^3 - 3(\tan 60^o)z_R{}^2 - 3z_R + (\tan 60^o) = 0$$

$$z_R{}^3 - 3\sqrt{3}z_R{}^2 - 3z_R + \sqrt{3} = 0$$

Or,

$$z_R{}^3 = 3\sqrt{3}z_R{}^2 + 3z_R - \sqrt{3}$$

Next, the *coefficients* for the second, independent *GCE* for $R=1$ established above are determined as follows:

Where,

$$3\theta = 60^o$$

$$\theta = 20^o$$

$$z_R = \tan\theta = 0.363970234$$

$$\beta = -(\frac{5\tan\theta - \tan^3\theta}{1-\tan^2\theta}) = -2.042169497 \qquad \textit{[See above]}$$

$$\gamma = \tan\theta(\frac{4\tan\theta}{1-\tan^2\theta}) - 1 = -0.389185421 \qquad \textit{[See above]}$$

$$\delta = \tan\theta = 0.363970234 \qquad \textit{[See above]}$$

Hence, this particular associated second, independent *Generalized Cubic Equation* for $R=1$ and $\alpha=1$ is:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 - 2.042169497z^2 - 0.389185421z + 0.363970234 = 0$$

Notice that for such equation:

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{0.363970234 - (-2.042169497)}{1-(-0.389185421)}$$

$$= \frac{2.406139731}{1.389185421}$$

$$\tan 60^o = \sqrt{3}$$

In other words, it possesses the *same value of ζ* as expressed in the *given 3θ Cubic Equation*.

As such, the result obtained during the *derivation* of *Equation 51 is applied as follows:*

$$(3\zeta+\beta)z_R{}^2+(3+\gamma)z_R+(\delta-\zeta)=0$$

$$(3\sqrt{3}-2.042169497)z_R{}^2+(3-0.389185421)z_R+(0.363970234-\sqrt{3})=0$$

$$3.153982926z_R{}^2+2.610814579z_R-1.368080573=0$$

Where,

$$z_1;z_2=\frac{1}{2a}(-b\pm\sqrt{b^2-4ac})$$

$$=\frac{1}{2(3.153982926)}[-2.610814579\pm\sqrt{(2.610814579)^2+4(3.153982926)(1.368080573)}]$$

$$=\frac{1}{2(3.153982926)}(-2.610814579\pm\sqrt{6.816352766+17.25961108})$$

$$=\frac{1}{6.307965852}(-2.610814579\pm4.906726388)$$

$$=0.363970234;-1.191753593$$

$$=\tan 20^o;-\frac{1}{\tan 40^o}$$

$$=\tan\theta;-\frac{1}{\tan(2\theta)}$$

In summary, Equation 51 depicts a ***remarkable portrayal*** of the very manner in which an *unknown* **common root z_R** manifests itself via inextricable linkage to *modifying coefficients*.

Other *well known* equation formats which relate *root structures* to their *coefficients* are as indicated below:

- The *Quadratic Formula* relates its *roots* to *respective coefficients* via the *Quadratic Formula* as follows:

  Where $ax^2+bx+c=0$,

  $$x_1;x_2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- The *Generalized Cubic Equation* relates its *roots* to *respective coefficients* as follows when $\beta^2=3\alpha\gamma$ and $\alpha=1$:

  $$z_R=\frac{-\beta+\sqrt[3]{\beta^3-27\delta}}{3}$$ *[Ref. Section 13.2]*

In conclusion, such *associations* between *equation coefficients* and their *intrinsic root structures* are best *characterized* by *mathematical interpretations* of their inherent ***RST Spreads.***

SECTION 20. *THE IMPOSSIBILITY OF EUCLIDEAN TRISECTION.*

To reiterate what clearly has been asserted many times over during the past years: An angle most certainly <u>cannot</u> be trisected *solely* via *Euclidean means*!

More specifically stated, that is to say it is ***impossible*** to trisect an angle, no matter what its size, when only a straightedge and compass are permitted to act upon it.

In response to the caveat that certain angles can be trisected, let it be said that such actions <u>cannot</u> be achieved solely by Euclidean means, but only when otherwise introducing ***extraneous*** information into such famous trisection problem, thereby corrupting it and, in so doing, enabling entirely different problem types to become solved.

In this sense, *extraneous information* is considered to entail any ***aforehand knowledge*** which can be derived from either algebraic determinations, or geometric applications other than those where a straightedge and compass become applied to an angle of given magnitude.

To continue, consider a ninety degree angle which is to be trisected by way of only a straightedge and compass.

Such process would consist of dividing such given magnitude by a factor of 3 (algebraically expressed as $90^{\circ}/3 = 30^{\circ}$), and, from the information gained, therafter geometrically constructing such trisector merely by means of bisecting any internal angle contained within a geometrically constructed equilateral triangle.

The only problem with such approach would be that ***aforehand knowledge*** would have to be introduced as to the precise magnitude of such sought after thirty degree trisector, well before such Euclidean process even would be given the opportunity to determine it, thereby enabling its eventual geometrical construction by a completely different method.

20.1. *Equation 51* Algebraic Determination of a Trisector is not Euclidean.

Next, consider the determination of a trisector by means of developing of a *second, independent Generalized Cubic Equation (GCE)* for $R = 1$ which turns out to be <u>devoid</u> of its *second* and *third* terms.

Such *second, independent GCE* very easily could be developed by means of setting its second and third term coefficients equal to zero; whereby β and γ would equal zero, and *Equation 51* would reveal that $z_R = -\zeta$ calculated as follows:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma} \qquad \text{[Ref.Equation 51]}$$

$$= \frac{3\zeta(0-1)-4(0)}{3+0}$$

$$= -\zeta$$

In this particular case, it not even is necessary to determine the value of the remaining *unknown coefficient*, *δ*, simply because the ***common root*** *value* z_R may be determined without it as follows:

$$\zeta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = \frac{3z_R - z_R{}^3}{1-3z_R{}^2} = -z_R \qquad \text{[Ref. Equation 3]}$$

$$z_R(3-z_R{}^2) = z_R(3z_R{}^2-1)$$

$$3-z_R{}^2 = 3z_R{}^2-1$$

$$-4z_R{}^2 = -4$$

$$z_R = \sqrt{1} = R\tan\theta = (1)\tan\theta = \pm 1 = \tan\theta$$

$$\arctan(+1);\arctan(-1) = \arctan\theta_1;\arctan\theta_2$$

$$45^o;135^o = \theta_1;\theta_2$$

Then,

$$3(45^o);3(135^o) = 3\theta_1;3\theta_2$$

$$135^o;405^o = 3\theta_1;3\theta_2$$

$$\tan 135^o;\tan(45+360)^o = \zeta_1;\zeta_2$$

Or,

$$\mp 1 = \zeta$$

$$\pm 1 = -\zeta = z_R$$

The beauty of this analysis is that *both* the ***common root*** $z_R = \pm 1$, and $\zeta = \tan(3\theta) = \mp 1$ could be *geometrically constructed* directly from a *rational number* of unity. More importantly, the angle θ *then could be **geometrically constructed** as a true* trisector of a given angle 3θ whose magnitude is either 135°, or 405° = (360° + 45°) = 45°.

Such result evidences that in order to trisect an angle of either 135° or 45° magnitude, all that is necessary is to geometrically construct a trisector of size 45° or 135°, respectively.

Please be apprised, however, that such conclusion could not be reached when otherwise attempting to trisect an angle of 135° magnitude solely by Euclidean means, nor even when attempting to trisect an angle whose magnitude is 45°.

20.2. A Repeated Bisections Approach to Perform Trisection.

A *series of bisections* contrived purely by *Euclidean construction operations* can be applied to achieve the *trisection* of *any given angle* solely via *compass* and *straightedge*.

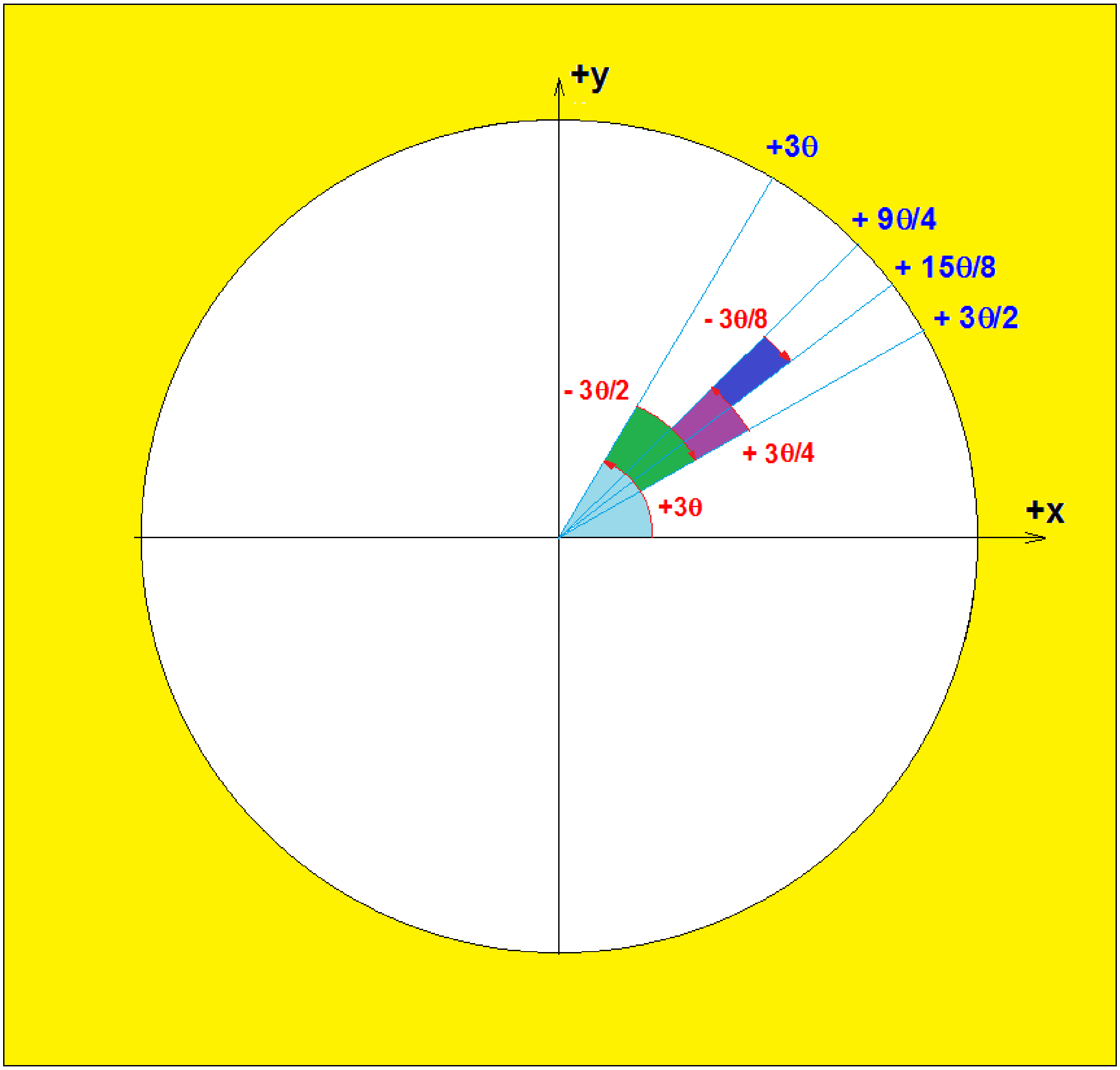
Its single drawback is that it requires an infinite *number of iterations* in order to produce an exact solution. As indicated below, *twenty such iterations* results in an accuracy of better than one in a million.

Such approach pertains to my previsouly unpublished *1994 American copyright number TXu 636-519*; as well to this *renewed 2014 copyright* reiteration, stating that:

> Where a *geometric progression* is a series of terms connected by a constant multiplier: For an *infinite number of terms*, its *sum*, hereinafter to be designated as "s" is based upon a relationship between its first term, "f", and a *common ratio* between its terms, "m", as follows:
>
> $$s = \frac{f}{1-m}$$ *FOOTNOTE* ***2***
>
> When its first term is set equal to 3θ, and m is equal to -1/2, "s" is found to be equal to 2θ as follows:
>
> $$s = \frac{3\theta}{1-(-1/2)} = \frac{3\theta}{3/2} = 2\theta$$
>
> Such result evidences that a given angle of magnitude 3θ can be trisected by a *series of Euclidean bisections* conducted in a certain sequence, as specified in *Figure 48*.

1. Section entitled *By Infinite Repetition of Bisection* as stipulated in Website: http://en.wikipedia.org/wiki/Angle_trisection
2. CRC Standard Mathematical Tables Twelfth Edition; The Chemical Rubber Co. Cleveland, OH; January 1964; page 357.

Figure 48. Bisecting an Angle an Infinite Number of Times in Orderly Fashion.



The *geometric progression* associated with such configuration is determined by constantly multiplying each successive term by −1/2. For a *geometric series* consisting of "n" terms, this may be expressed as:

$$s = 3\theta - \frac{3\theta}{2} + \frac{3\theta}{4} - \frac{3\theta}{8} + \frac{3\theta}{2^{n-2}} - \frac{3\theta}{2^{n-1}}$$

The first four of such designated terms is located within the circle illustrated in *Figure 48*. They represent *swings* of specified angles from a given *start point* where counterclockwise movement is notated by a positive swing. The location of each respective *end point* is identified outside of the circle. Each location represents a summation of the above specified *geometric progression* for the quantity of terms being depicted. Such respective calculations are afforded in *Table 34*.

Table 34. Trisection Accuracy Plot.

n	SUMMATION	TOTAL
1	3θ	3.0000000θ
2	$3\theta - 3\theta/2 = 3\theta/2$	1.5000000θ
3	$3\theta/2 + 3\theta/4 = 9\theta/4$	2.2500000θ
4	$9\theta/4 - 3\theta/8 = 15\theta/8$	1.8750000θ
5	$15\theta/8 + 3\theta/16 = 33\theta/16$	2.0625000θ
6	$33\theta/16 - 3\theta/32 = 63\theta/32$	1.9687500θ
7	$63\theta/32 + 3\theta/64 = 129\theta/64$	2.0156300θ
8	$129\theta/64 - 3\theta/12/ = 255\theta/128$	1.9921900θ
9	$255\theta/128 + 3\theta/256 = 513\theta/256$	2.0039100θ
10	$513\theta/256 - 3\theta/512 = 1023\theta/512$	1.9980500θ
11	$1023\theta/512 + 3\theta/1024 = 2049\theta/1024$	2.0009800θ
12	$2049\theta/1024 - 3\theta/2048 = 4095\theta/2048$	1.9995100θ
13	$4095\theta/2048 + 3\theta/4096 = 8193\theta/4096$	2.0002400θ
14	$8193\theta/4096 - 3\theta/8192 = 16383\theta/8192$	1.9998800θ
15	$16383\theta/8192 + 3\theta/16384 = 32769\theta/16384$	2.0000600θ
16	$32769\theta/16384 - 3\theta/32768 = 65535\theta/32768$	1.9999700θ
17	$65535\theta/32768 + 3\theta/65536 = 131073\theta/65536$	2.0000200θ
18	$131073\theta/65536 - 3\theta/131072 = 262143\theta/131072$	1.9999900θ
19	$262143\theta/131072 + 3\theta/262144 = 524289\theta/262144$	2.0000000θ
20	$524289\theta/262144 - 3\theta/524288 = 1048575\theta/524288$	2.0000000θ

Table 34 reports an accuracy for 2θ of 1/100 at n equals 7, and better than 1/1,000,000 at n equals twenty iterations.

In order to determine the *true trisector value*, θ, this resultant 2θ value may be *bisected*, or it may be subtracted from the 3θ value stipulated outside of the circle in *Figure 48.*

The only *seeming* limitation which might be accorded to the *repeated bisection* construction process is that exact, or complete, trisection *theoretically* can occur only after an *infinite number of operations* -- thereby qualifying as an activity that would take more than a human lifetime to complete, and hence being considered to be a ***physical impossibility***; whereby such approach would remain consistent with the assertion that an angle cannot be trisected solely by Euclidean means.

20.3. The Impossibility of Attempting to Geometrically Construct *Equation 1*.

Although <u>all</u> *coefficients* appearing in *Equation 1* are of *rational* value, its *root sets* can be *cubic irrational*.

For example when,

$3\theta = 60°$

$\cos(3\theta) = 1/2$

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos(3\theta)$$ *[Ref. Equation 1]*

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}(1/2)$$

$$4\cos^3\theta = 3\cos\theta + 1/2$$

Whereby,

$\theta_1 = 60°/3$

$\theta_1 = 20°$

Such that the root set $(\cos\theta_1, \cos\theta_2, \cos\theta_3)$ assumes all *irrational values* as follows:

$\cos\theta_1 = \cos 20° = 0.93969262\ldots$

$\cos\theta_2 = \cos 140° = -0.766044443\ldots$, and

$\cos\theta_3 = \cos 260° = -0.173648177\ldots$

check

$$\cos^3\theta_1 = \frac{3}{4}\cos\theta_1 + \frac{1}{4}(1/2)$$

$$4(0.93969262...)^3 = 3(0.93969262...) + 1/2$$

$$3.319077862... = 2.819077862... + 1/2$$

$$= 3.319077862... \quad \surd$$

$$\cos^3\theta_2 = \frac{3}{4}\cos\theta_2 + \frac{1}{4}(1/2)$$

$$-1.798133329\ldots = -2.298133329\ldots + 1/2$$

$$= -1.798133329\ldots \quad \surd$$

$$\cos^3\theta_3 = \frac{3}{4}\cos\theta_3 + \frac{1}{4}(1/2)$$

$$-0.020944533\ldots = -0.520944533\ldots + 1/2$$

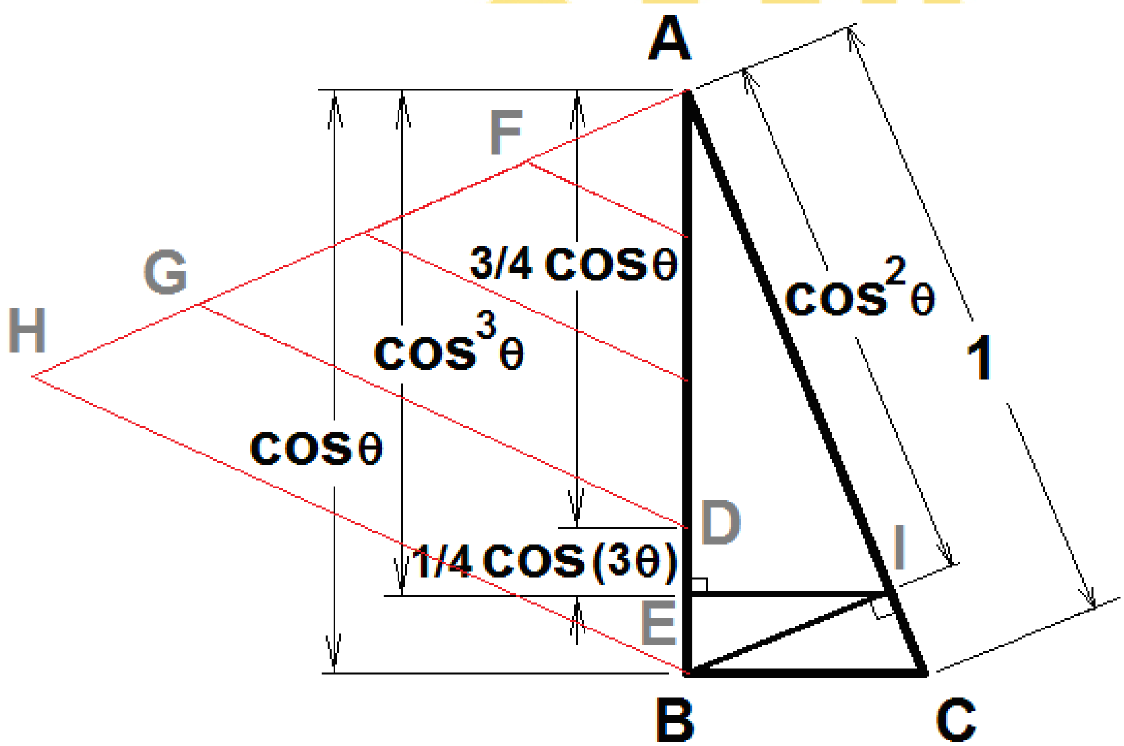
$$= -0.020944533\ldots \quad \surd$$

In conjunction with such kind of result, *Figure 49* is to become introduced such that it depicts a right triangle ABC whose hypotenuse is constructed equal to one unit of measurement.

Specific constraints imposed upon it by *Equation 1* then are to be as follows:

- Along line $\overline{AB}$, the length $\cos^3\theta$ must be *equal* to the sum of the lengths ¾ $\cos\theta$ and ¼ $\cos(3\theta) = \tau/4$;
- Along line $\overline{AC}$, the length $\cos^2\theta$, as drawn from point A, must *terminate* at a perpendicular *dropped* to *such line* from point B;
- Along line $\overline{AB}$, a perpendicular drawn from the termination point of length $\cos^3\theta$ must intersect the *termination point* of length $\cos^2\theta$ which was previously identified along line $\overline{AC}$; and
- Along line $\overline{AB}$, the length ¾ $\cos\theta$ terminates at the point obtained after bisecting *such line* twice.

Figure 49. Construction to Determine cos θ from any Supplied ¼ cos (3θ) Value.



Such geometric construction can be achieved only by *trial and error* as follows:

a) $\tau/4 = \cos(3\theta)/4 = 0.5/4 = 1/8$ is to be selected, and thereafter *geometrically constructed* as straight line length $\overline{DE}$;

b) Such line then is extended to point A, by an ***unknown, arbitrary length*** labeled as $(3/4)\cos\theta$;

c) Another line to be labeled as $\overline{AB}$ then is *geometrically constructed* of length equal to 4/3 that of the newly constituted line $\overline{AD}$. Such length is *geometrically constructed* by drawing straight line $\overline{AF}$ of any arbitrary length and angle away from line $\overline{AE}$. This is repeated along such straight line three consecutive times, thereby comprising line $\overline{AG}$. Straight line $\overline{DG}$ is thereafter drawn. As illustrated, lines drawn parallel to $\overline{DG}$ located at distances $\overline{AF}$ apart along line $\overline{AG}$ will divide line $\overline{AD}$ into three equal segments. Hence, extending line $\overline{AG}$ again by the length $\overline{AF}$ locates point H. Another line drawn parallel to $\overline{DG}$ which goes through point H intersects line $\overline{AE}$ extended at point B. Then, via similar triangles:

$$\overline{AG}/\overline{AH} = 3/4 = \overline{AD}/\overline{AB}$$

$$\overline{AB} = (4/3)\overline{AD} = (4/3)(3/4)\cos\theta = \cos\theta$$

d) Next straight lines are drawn perpendicular to line $\overline{AB}$ which emanate from points B and E, respectively;

e) Point C then is located by drawing a radius of *one unit in length* from point A and noting where it intersects such perpendicular line drawn to line $\overline{AB}$ emanating from point B;

f) Point I represents the intersection point between line $\overline{AC}$ and such perpendicular line drawn to line $\overline{AB}$ emanating from point E;

g) From point I, a line is drawn perpendicular to line $\overline{AC}$

h) If such new line travels through point B, then line $\overline{AE}$ is equal to $\cos^3\theta$.

i) If not, then the whole process is repeated by adjusting the arbitrary length $\overline{AD} = (3/4)\cos\theta$.

Notice that the equality provided above can be rearranged as follows:

$$4\cos^3\theta = 3\cos\theta + 1/2$$

$$8\cos^3\theta - 6\cos\theta - 1 = 0$$

It should be clearly understood that it would be virtually impossible to obtain an exact geometric solution from such *trial and error process* because it would take an *infinite number* of purely *random trial and error* iterations before the exact length of $\cos^3\theta$ could be selected. Naturally, such contention does not consider a ***refinement process*** whereby the results of each *geometric construction* step could be used to constitute a starting point for the next).

Likewise, it furthermore remains impossible to ***geometrically construct*** a *cubic irrational length* from *a rational* one. Specifically, this is because:

What is <u>not</u> *rationally-based* is *cubic irrational* and it is impossible to pronounce and represent its value via *geometric construction*.

And so, it is concluded that attempting to ***geometrically construct*** a *straight line* of *cubic irrational* length from any combination of *rationally-based lengths* is clearly *impossible (Ref. Section 9.1)*.

Moreover, the reason why a *Generalized Cubic Equation* with completely ***rational coefficients*** which assumes the *format* of *Equation 1* can be developed that accounts for any selected *rational root value* is because when the summation of its left-hand terms becomes *rational*, then so must its right-hand term which is equal to $\cos(3\varphi)$.

$$4\cos^3\phi - 3\cos\phi = \cos(3\phi) \qquad \textit{[Ref. Equation 1]}$$

So, for the specific condition when $\cos\varphi = 1/3$:

$$4(\frac{1}{3})^3 - 3(\frac{1}{3}) = \cos(3\phi)$$

$$\frac{4}{27} - \frac{27}{27} = \cos(3\phi)$$

$$-\frac{23}{27} = \cos(3\phi)$$

Check,

$$-23/27 = \cos(3\phi)$$

$$148.4136619^o; 211.5963381^o = 3\phi$$

$$49.47122063^o; 70.52877937^o = \phi$$

$$0.649829914; \frac{1}{3} = \cos\phi \qquad \text{Q.E.D.}$$

As such, the following equation possesses *rational coefficients*, a *rational root*, and belongs to the *Equation 1 curve family*:

$$4\cos^3\phi - 3\cos\phi + \frac{23}{27} = 0$$

$$\cos^3\phi - \frac{3}{4}\cos\phi + \frac{23}{108} = 0$$

Moreover, no respective root may be characterized for a value of $R = 1$, demonstrated as follows:

$$z_R = \tan\theta_R = 1/3 = \cos 70.52877937^o$$

$$z_S = \tan\theta_S = \cos(\theta_R + 120^o) = \cos 190.52877937^o = -0.983163247$$

$$z_T = \tan\theta_T = \cos(\theta_R + 240^o) = \cos 310.52877937^o = 0.649829914$$

$$\theta_R = \arctan z_R = \arctan(1/3) = 18.43494882^o$$

$$\theta_S = \arctan z_S = \arctan(-0.983163247) = -44.51357929^o$$

$$\theta_T = \arctan z_T = \arctan(0.649829914) = \underline{33.01701627^o}$$

$$\Sigma = 3\theta + 360^o = 3\theta = 6.938385804^o$$

Where,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{23/108 - 0}{1 - (-3/4)}$$

$$\tan(3\theta) = \frac{23}{7(27)}$$

$$3\theta = 6.938385804^o \qquad \text{Q.E.D.}$$

$$\theta = 2.312795298^o$$

$$\tan\theta = 0.04038783$$

$$z_R = R\tan\theta = 0.04038783R = 1/3$$

$$z_R = S\tan\theta = 0.04038783S = -0.983163247$$

$$z_R = T\tan\theta = 0.04038783T = 0.649829914$$

$$R = \frac{1}{3(0.04038783)} \neq 1$$
$$S = \frac{-0.983163247}{0.04038783} \neq 1$$
$$T = \frac{0.649829914}{0.04038783} \neq 1$$

Accordingly, the *established Cubic Equation* exhibits added complexity, whereby *Equation 51* cannot be invoked.

Nevertheless, its *rational root* $z_R = 1/3$ still cannot be *geometrically constructed* from an *arrangement* of its *rational coefficients*; unless, of course such *root* becomes either:

a) Afforded *aforehand (Ref. Section 19)*;

b) *Randomly selected* from the *infinite number* of *rational values* available in the *rational field* F_0; or

c) Discernable as the result of a *determinable second, independent Generalized Cubic Equation (Ref. Section 19)*.

SECTION 21. *CUBE ROOT CONSIDERATIONS.*

The *well-known Quadratic Formula* (see below) affirms that *root set values* belonging to ***Quadratic Equations*** of the form $ax^2 + bx + c = 0$ can be *algebraically determined* solely from their *coefficient structures.*

$$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Of note, they furthermore can be ***geometrically constructed*** by means of performing the *Euclidean mapping procedure* stipulated in Section 2.3; whereby the values of their *coefficients* would become represented by lengths of given size.

Some *mathematicians*, upon becoming inspired by such ***coefficient driven*** realization, naturally might try to identify some hidden, unknown ***inextricable geometric linkage*** that could associate solely ***rational coefficients*** inherent within *Generalized Cubic Equation formats* to their intrinsic ***cubic irrational root set*** counterparts.

Obviously, such type of breakthrough might even unlock the mystery of how to divide a given angle of unknown size into three equal parts when acting upon it only by means of applying a *straightedge and compass*; and in so doing, accomplishing a *Euclidean trisection*.

In order to achieve such goal, it appears that various methods were resorted to in the past which relied upon attempts to *geometrically construct* ***cube roots***.

The association that ***Equation sub-element theory*** bears upon such ***cube roots*** phenomenon is presented below. In many cases, algebraic interpretations are supplied, thereby becoming disqualifying as methods which could accomplish *Euclidean trisection*.

a) **Explaining why attempting to *geometrically construct cube roots* is synonymous with trisection, and therefore *cannot* be achieved solely by *Euclidean means*:**
   With regard to the factor $\cos(2\omega)$, as contained in the variable ℓ of the *Cubic Resolution Transform (CRT) presented* below, an association with ***cube roots*** can be established as follows *(Ref. Section 13.3)*:

$$f^3 \pm \left(\frac{3\ell}{2\psi}\right) f^2 \mp \left(\frac{\ell^3}{2\psi}\right) = 0$$ *[Ref. Equation 38]*

Such that

$\ell = 2f\cos(2\omega)$ *[Ref. Figure 11]*

Where the formula for a *Binomial Expansion* of the *cube* of the *polynomial* $A \pm B$ is as follows:

$$(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$$

For the specific circumstance when:

$$A = \cos(2\omega)$$
$$B = i\sin(2\omega)$$

$$\begin{aligned}(A \pm B)^3 &= [\cos(2\omega)]^3 \pm 3[\cos(2\omega)]^2[i\sin(2\omega)] + 3[\cos(2\omega)][i\sin(2\omega)]^2 \pm [i\sin(2\omega)]^3 \\ &= \cos^3(2\omega) \pm 3[1-\sin^2(2\omega)][i\sin(2\omega)] - 3[\cos(2\omega)][1-\cos^2(2\omega)] \mp i\sin^3(2\omega) \\ &= [4\cos^3(2\omega) - 3\cos(2\omega)] \pm i[3\sin(2\omega) - 4\sin^3(2\omega)] \\ &= \cos(6\omega) \pm i\sin(6\omega)\end{aligned}$$

Taking the *cube root* of each side affords:

$$A + B = \cos(2\omega) + i\sin(2\omega) = \sqrt[3]{\cos(6\omega) + i\sin(6\omega)}$$
$$A - B = \cos(2\omega) - i\sin(2\omega) = \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}$$

Such that by summing the two above equations,

$$2\cos(2\omega) = \sqrt[3]{\cos(6\omega) + i\sin(6\omega)} + \sqrt[3]{\cos(6\omega) - i\sin(6\omega)}$$

Now, upon letting ψ represent $\cos(6\omega)$, the following equality can be established,

$$\cos^2(6\omega) + \sin^2(6\omega) = 1$$
$$\psi^2 + \sin^2(6\omega) = 1$$
$$\sin(6\omega) = \sqrt{1-\psi^2}$$

Then, by substituting this result into the equation above, it can be shown that,

$$\begin{aligned}2\cos(2\omega) &= \sqrt[3]{\psi + i\sqrt{1-\psi^2}} + \sqrt[3]{\psi - i\sqrt{1-\psi^2}} \\ &= \sqrt[3]{\psi + i\sqrt{(-1)(\psi^2-1)}} + \sqrt[3]{\psi - i\sqrt{(-1)(\psi^2-1)}} \\ &= \sqrt[3]{\psi + i^2\sqrt{\psi^2-1)}} + \sqrt[3]{\psi - i^2\sqrt{\psi^2-1)}} \\ &= \sqrt[3]{\psi - \sqrt{\psi^2-1)}} + \sqrt[3]{\psi + \sqrt{\psi^2-1)}}\end{aligned}$$

Since real values for ψ exist within the range from -1 to +1, then the radical $\sqrt{\psi^2-1}$

must be *imaginary* or equal to zero. Hence, except for such latter case, each of the terms appearing under the two *cube root radicals* indicated above must be *complex numbers*. Now, since taking the *cube root* of a complex number is synonymous with representing its trisector in a Cartesian Coordinate System, it would appear to be impossible to *geometrically construct* it solely by *Euclidean means*.

b) Showing how *cube roots* can be eliminated through algebraic manipulation:

Except for certain *very rare instances (Ref. Section 20)*, an *unknown quantity z* may be represented as the *negative* ***cube root*** of the summation of *second*, *third* and *fourth terms* of a given *Generalized Cubic Equation* for $\alpha = 1$ that becomes mathematically reorganized as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

$$z^3 = -\beta z^2 - \gamma z - \delta$$

$$= (-1)(\beta z^2 + \gamma z + \delta)$$

$$= (-1)^3(\beta z^2 + \gamma z + \delta)$$

$$z = -\sqrt[3]{\beta z^2 + \gamma z + \delta}$$

Since such 2^{nd} *and* 3^{rd} *terms* include the *unknown root, z*, its value is ***required aforehand*** in order to determine the value of the left-hand side of the above equation. Hence, such algebraic relationship cannot contribute towards attempting to trisect an angle solely by Euclidean means *(Ref. Section 19)*.

1) For rational values of z_R *and* ζ when R=1 and $\beta=0$:

Interposing rational values of $z_R = \tan\theta$ and $\zeta = \tan(3\theta)$ into the *3θ Cubic Equation* enables results to be obtained which thereafter could be geometrically constructed, as based upon such input. For example, when $z_R = 1/3$ *(Ref. Section 19 Example)*:

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0 \qquad \textit{[3θ Cubic Equation]}$$

$$(1/3)^3 - 3\zeta(1/3)^2 - 3(1/3) + \zeta = 0$$

$$1/27 + \zeta(1 - 1/3) - 1 = 0$$

$$\zeta(18/27) = 26/27$$

$$\zeta = 13/9$$

A *second, independent Generalized Cubic Equation (GCE)* for $R=1$ and $\beta=0$ can be determined as:

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$1/3 = \frac{3(13/9)(\gamma-1) - 4(0)}{3+\gamma}$$

$$(1/3)(3+\gamma) = (13/3)(\gamma-1)$$

$$3+\gamma = 13\gamma - 13$$

$$16 = 12\gamma$$

$$4/3 = \gamma$$

Hence, the two above determined equations can be combined in order to be *resolved* simultaneously via the *Quadratic Formula*, or the ***geometric construction*** *Mapping Process* presented in *Section 2.3*, as follows:

$$z_R^{\ 3} - 3\zeta z_R^{\ 2} - 3z_R + \zeta = 0$$

$$z_R^{\ 3} = 3\zeta z_R^{\ 2} + 3z_R - \zeta$$

For $\alpha = 1$

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \qquad \textit{[Ref. Equation 32]}$$

$$z^3 + \beta z^2 + \gamma z + \delta = 0$$

Via substitution from above:

$$[3\zeta z_R^{\ 2} + 3z_R - \zeta] + \beta z_R^{\ 2} + \gamma z_R + \delta = 0$$

$$(3\zeta + \beta)z_R^{\ 2} + (3+\gamma)z_R + (\delta - \zeta) = 0$$

$$z_R = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textit{[Ref. Quadratic Formula]}$$

$$z_R = \frac{-(3+\gamma) \pm \sqrt{(3+\gamma)^2 - 4(3\zeta+\beta)(\delta-\zeta)}}{2(3\zeta+\beta)}$$

$$= \frac{-(3+4/3) \pm \sqrt{(13/3)^2 - 4(13/3+0)[\zeta(1-\gamma)+\beta-\zeta]}}{2(13/3+0)}$$

$$= \frac{-(13/3) \pm \sqrt{(169/9) + 4(13/3)(\zeta\gamma - \beta)]}}{2(13/3)}$$

$$= \frac{-(13/3) \pm \sqrt{(169/9) + 4(13/3)(13/9)(4/3)}}{26/3}$$

$$= \frac{-(13/3) \pm (1/3)\sqrt{169 + 4(13/3)(13)(4/3)}}{26/3}$$

$$= \frac{-13 \pm 13\sqrt{(9+16)/9}}{26}$$

$$= \frac{-1 \pm 5/3}{2}$$

$$= 1/3; -4/3$$

Accordingly:

- From a *given angle* $3\theta = 55.30484647^o$, $\zeta = \tan(3\theta) = 13/9$ can be ***geometrically constructed***
- From the synthesis of such *two equations*, a common root $z_R = \tan\theta = 1/3$ can be ***geometrically constructed*** using the Quadratic Equation expressed above via the mapping process stipulated in *Section 2.3*
- From such ***geometrically constructed*** length of $z_R = \tan\theta = 1/3$, an *angle* θ then can be ***geometrically constructed*** which is equal to 18.43494882^o, or exactly 1/3 the magnitude of such *given angle* $3\theta = 55.30484647^o$. Since such geometric construction relies upon the results of an algebraic analysis as *aforehand knowledge*, such process does not qualify as a valid ***Euclidean*** trisection

Above, notice that it is not necessary to extract a ***cube root*** in order to algebraically determine such solution.

2) For $\beta=\gamma=0$:

An associated analysis begins by examining the *Generalized Cubic Equation* for conditions when $\alpha = 1$ as follows:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0 \quad \textit{[Ref. Equation 32]}$$

$$z^3 = -(\beta z^2 + \gamma z + \delta)$$

$$z = -\sqrt[3]{\beta z^2 + \gamma z + \delta}$$

Notice above that in order to calculate a ***root z***, it first becomes necessary to extract the ***cube root*** of a value which is comprised of *multiples* and *mathematical combinations* of such <u>unknown</u> *quantity*.

However, this doesn't apply when $\beta = \gamma = 0$ as follows:

$$z^3 + (0)z^2 + (0)z + \delta = 0$$

$$z^3 + \delta = 0 \quad \textit{[Ref. Section 13.5]}$$

Where,

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \quad \textit{[Ref. Equation 36]}$$

$$= \frac{\delta - 0}{1 - 0}$$

$$= \delta$$

Via substitution:

$$z_R^{\ 3} + \delta = 0$$

$$z_R^{\ 3} + \zeta = 0$$

$$(R \tan\theta)^3 + \zeta = 0$$

When $R = 1$, the above equation then relates $\tan\theta$ to $\zeta = \tan(3\theta)$ where,

- $\zeta = \tan(3\theta)$ is a value which can be *geometrically constructed* from any ***given angle 3θ***
- $\tan\theta$ is a value from which ***trisected angle θ*** can be *geometrically constructed*

Under such conditions,

$$\tan^3\theta + \zeta = 0$$

Via further substitution of *Equation 3*:

$$\tan^3\theta + \frac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta} = 0$$

$$\tan^2\theta + \frac{(3-\tan^2\theta)}{1-3\tan^2\theta} = 0$$

$$\tan^2\theta(1-3\tan^2\theta) + 3 - \tan^2\theta = 0$$

$$-3\tan^4\theta + 3 = 0$$

$$-\tan^4\theta + 1 = 0$$

$$1 = \tan^4\theta$$

$$\pm 1 = \tan^2\theta$$

$$\pm 1; i = \tan\theta$$

$$\pm 45^o = \theta$$

$$45^o; 315^o = \theta$$

$$135^o; 945^o = 3\theta$$

$$135^o; 225^o = 3\theta$$

$$\mp 1 = \zeta$$

Accordingly,

$$\tan^3\theta + \zeta = 0$$

$$\tan^3\theta \mp 1 = 0$$

$$\tan\theta = \pm\sqrt[3]{1}$$

$$\tan\theta = \pm 1$$

Since the ***cube root*** of *unity* is defined as *unity*, an algebraic solution becomes afforded without having to extract such ***cube root***.

This above *finding* is independently confirmed by *Equation 51* which applies because $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$ as follows:

$$z_R = \frac{3\zeta(\gamma-1)-4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$= \frac{3\zeta(0-1)-4(0)}{3+0}$$

$$= -\zeta$$ *[Ref. Section 20.1]*

$$= {z_R}^3$$ *[Since ${z_R}^3 + \zeta = 0$ above]*

$$1 = {z_R}^2$$

Taking the *square root* produces values for z_R as follows:

$$\sqrt{1} = z_R$$

$$\pm 1 = z_R \quad \text{[Ref. Section 20.1]}$$

$$= \tan\theta$$

$$\arctan(\pm 1) = \theta$$

$$45^o ; 135^o = \theta$$

$$135^o ; 45^o = 3\theta$$

$$\tan 135^o ; \tan 45^o = \tan(3\theta)$$

$$-1 ; +1 = \zeta$$

Check,

$z_R{}^3 + \zeta = 0$	$z_R{}^3 + \zeta = 0$
$z_R{}^3 - 1 = 0$	$z_R{}^3 + 1 = 0$
$1^3 - 1 = 0$	$(-1)^3 + 1 = 0$
$1 - 1 = 0$	$-1 + 1 = 0$
$0 = 0$	$0 = 0$

As such, the two specifically determined *Generalized Cubic Equations*, $z_R{}^3 = \pm 1$, do not require ***cube roots*** to be *geometrically constructed* because they can be reduced to respective *Quadratic Equations* as demonstrated above.

3) **For Circumstances when *Generalized Cubic Equations* exhibit coefficients in prescribed ratios:**
By now, it should be realized that conducting *geometric construction* upon any *given value of* $\zeta = \tan(3\theta)$ is far different than *geometrically assessing* ***coefficients*** which belong to an *associated Cubic Equation*. Moreover, this distinction applies even when such *coefficients* just so happen to be ***irrational***.

This is because *algebraic assessment* and *geometric construction* are ***far different*** entities. So, it seems fitting that they, indeed, are represented by *different branches* of *mathematics*.

And so it is that trisection can be algebraically determined far more readily from given *cubic equations* than solely from *given* geometric *values* of $\zeta = \tan(3\theta)$; despite the fact that such algebraic solutions cannot constitute *Euclidean* trisections!

Algebraic determinations of such types become accomplished simply by first *interpreting*, and thereafter *geometrically operating* upon the *coefficient structures* of *given Cubic Equations*.

Perhaps the example which is easiest to comprehend pertains to a ***cubic root*** which, in fact, is equal to a fraction of a coefficient which appears in a *Generalized Cubic Equation*.

For purposes of illustration, for:

$$\beta = -3z_R$$

$$-\frac{\beta}{3} = z_R$$

$$0 = z_R + \frac{\beta}{3}$$

The ***cube*** of the *above binomial* is:

$$0 = (z_R + \frac{\beta}{3})^3$$

$$= z_R{}^3 + 3(\beta/3)z_R{}^2 + 3(\beta/3)^2 z_R + (\beta/3)^3$$

$$= z_R{}^3 + \beta z_R{}^2 + (\beta^2/3)z_R + \beta^3/27$$

Such that,

$$0 = \alpha z^3 + \beta z^2 + \gamma z + \delta \qquad \textit{[Ref. Equation 32]}$$

Matching *like coefficients* renders:

$$\alpha = 1$$

$$\gamma = \beta^2/3$$

$$\delta = \beta^3/27$$

As such, a *Generalized Cubic Equation* whose ***coefficients*** appear in the *respective proportions* afforded below contains a *root* equal to $z_R = -\beta/3$:

$$z_R{}^3 + \beta z_R{}^2 + (\beta^2/3)z_R + \beta^3/27 = 0$$

Notice that for this above case, the value of the *coefficient* β can be either *rationally-based*, or *cubic irrational*.

The *geometric construction* aspect of this analysis becomes rudimentary since it consists simply of *geometrically* dividing any given value of β into three equal portions in order to determine the value of its associated *root* z_R.

Moreover, since $\beta^2 = 3\alpha\gamma = 3(1)\gamma = 3\gamma$, the following equation also applies *(Ref. Section 13.2)*:

$$z_R = R\tan\theta = \frac{-\beta + \sqrt[3]{\beta^3 - 27\alpha^2\delta}}{3\alpha}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(1)^2\delta}}{3(1)}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - 27(\beta^3/27)}}{3}$$
$$= \frac{-\beta + \sqrt[3]{\beta^3 - \beta^3}}{3}$$
$$= \frac{-\beta}{3}$$

However, in many cases note that $R \neq 1$.

As indicated above, the ***cube root*** term always adds out to zero when making use of such *Generalized Cubic Equation* ***format.***

Check,

$$\tan(3\theta) = \zeta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \quad \textit{[Ref. Equation 3]}$$
$$= \frac{z_R(3 - z_R{}^2)}{1 - 3z_R{}^2}$$
$$= \frac{-\frac{\beta}{3}[3 - (-\frac{\beta}{3})^2]}{1 - 3(-\frac{\beta}{3})^2}$$
$$= \frac{-\frac{\beta}{3}(3 - \frac{\beta^2}{9})}{1 - (\frac{\beta^2}{3})}$$
$$= \frac{\frac{\beta^3}{27} - \beta}{1 - (\frac{\beta^2}{3})}$$

$$= \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

Hence, by comparing like aspects of the above two equations:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *[Ref. Equation 32]*

$$z_R{}^3 + \beta z_R{}^2 + (\beta^2/3) z_R + \beta^3/27 = 0$$ *Q.E.D.*

Unfortunately, this *above analysis* represents *little more* than determining equations for any ***prescribed root*** z_R whose *coefficient β can be acted upon via* ***geometric* construction** for purposes of again identifying or producing such ***given root***.

Three other *Cubic Equations* of the above *format* are determined below through a *simplified process*. One exhibits a *rational cubic root*, another contains a *cubic root* comprised of a *square root quantity* that can be *geometrically constructed* via the mapping process specified in *Section 2.3*, and another expresses an *cubic irrational cubic root* as follows:

For

$$z_R = \tan\theta = 1/5$$

$$\beta = -3z_R$$
$$= -3/5$$

$$\gamma = \beta^2/3$$
$$= 3/25$$

$$\delta = \beta^3/27$$
$$= \gamma\beta/9$$
$$= -1/125$$

For

$$z_R = \tan\theta = 3 + \sqrt{7}$$

$$\beta = -3z_R$$
$$= -3(3 + \sqrt{7})$$

$$\gamma = \beta^2/3$$
$$= 3(16 + 6\sqrt{7})$$

$$\delta = \beta^3/27$$
$$= \gamma\beta/9$$
$$= -1(90 + 34\sqrt{7})$$

For

$$z_R = \tan\theta = \tan 20^o = 0.363970234$$

$$\beta = -3z_R$$
$$= -1.091910703$$

$$\gamma = \beta^2/3$$
$$= 0.397422994$$

$$\delta = \beta^3/27$$
$$= \gamma\beta/9$$
$$= -0.048216713$$

Check,

$$z^3 - \frac{3}{5}z^2 + (\frac{3}{25})z - \frac{1}{125} = 0$$

$$(\frac{1}{5})^3 - \frac{3}{5}(\frac{1}{5})^2 + (\frac{3}{25})(\frac{1}{5}) - \frac{1}{125} = 0$$

$$1 - 3 + 3 - 1 = 0$$

$$0 = 0$$

$$z^3 - 3(3+\sqrt{7})z^2 + 3(16+6\sqrt{7})z - (90+34\sqrt{7}) = 0$$

$$(3+\sqrt{7})^3 - 3(3+\sqrt{7})(3+\sqrt{7})^2 + 3(16+6\sqrt{7})(3+\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(27+27\sqrt{7}+63+7\sqrt{7}) - 3(3+\sqrt{7})(16+6\sqrt{7}) + 3(16+6\sqrt{7})(3+\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(90+34\sqrt{7}) + (3-3)(3+\sqrt{7})(16+6\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$(90+34\sqrt{7}) - (90+34\sqrt{7}) = 0$$

$$0 = 0$$

$$z^3 - 1.091910703z^2 + 0.397422994z - 0.048216713 = 0$$

$$(0.363970234)^3 - 1.091910703(0.363970234)^2 + 0.397422994(0.363970234) - 0.048216713 = 0$$

$$0.048216713 - 0.14465014 + 0.14465014 - 0.048216713 = 0$$

$$0 = 0$$

From these above determined Cubic Equations, roots may be determined ***linearly*** via the expression *posed in Equation 51 as follows:*

For $z^3 - \frac{3}{5}z^2 + (\frac{3}{25})z - \frac{1}{125} = 0$

Where,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \textit{[Ref. Equation 36]}$$

$$= \frac{-\frac{1}{125} + \frac{3}{5}(\frac{25}{25})}{\frac{125}{125} - \frac{3}{25}(\frac{5}{5})}$$

$$= \frac{74}{110}$$

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma} \qquad \textit{[Ref. Equation 51]}$$

$$= \frac{3(\frac{74}{110})[\frac{3}{25}(\frac{5}{5}) - (\frac{125}{125})] - 4(-\frac{3}{5})(\frac{25}{25})}{3(\frac{125}{125}) + \frac{3}{25}(\frac{5}{5})}$$

$$= \frac{3(\frac{74}{110})(-110) + 300}{375+15}$$

$$= \frac{78}{390}$$

$$= \frac{1}{5}$$

Q.E.D.

For $z^3 - 3(3+\sqrt{7})z^2 + 3(16+6\sqrt{7})z - (90+34\sqrt{7}) = 0$

Where,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \text{[Ref. Equation 36]}$$

$$= \frac{-(90+34\sqrt{7}) + 3(3+\sqrt{7})}{1-3(16+6\sqrt{7})}$$

$$= \frac{81+31\sqrt{7}}{47+18\sqrt{7}}$$

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma} \quad \text{[Ref. Equation 51]}$$

$$= \frac{3(\frac{81+31\sqrt{7}}{47+18\sqrt{7}})[3(16+6\sqrt{7})-1] + 12(3+\sqrt{7})}{3+3(16+6\sqrt{7})}$$

$$= \frac{(\frac{243+93\sqrt{7}}{47+18\sqrt{7}})(47+18\sqrt{7}) + 36 + 12\sqrt{7}}{51+18\sqrt{7}}$$

$$= \frac{279+105\sqrt{7}}{51+18\sqrt{7}}$$

$$= \frac{(3+\sqrt{7})(51+18\sqrt{7})}{51+18\sqrt{7}}$$

$$= 3+\sqrt{7} \qquad \text{Q.E.D.}$$

For

$z^3 - 1.091910703z^2 + 0.397422994z - 0.048216713 = 0$

Where,

$$\zeta = \frac{\delta - \beta}{1-\gamma} \qquad \text{[Ref. Equation 36]}$$

$$= \frac{-0.048216713 + 1.091910703}{1-0.397422994}$$

$$= \sqrt{3}$$

$$z_R = \frac{3\zeta(\gamma - 1) - 4\beta}{3 + \gamma}$$ *[Ref. Equation 51]*

$$= \frac{3\sqrt{3}(0.397422994 - 1) + 4(1.091910703)}{3 + 0.397422994}$$

$$= \frac{-3\sqrt{3}(0.602577005) + 4.367642811}{3.397422994}$$

$$= \frac{-3.131081968 + 4.367642811}{3.397422994}$$

$$= \frac{1.236560843}{3.397422994}$$

$$= 0.363970234$$ *Q.E.D.*

4) For Applications of the Trisector Equation Generator:

Naturally it is of far greater interest to derive an ***algorithm*** which instead determines equation types from given, or known values of $\zeta = \tan(3\theta)$ where their associated ***cube root*** terms also *add out to zero*.

This is accomplished as follows, where:

$$z_R = R\tan\theta = (1)\tan\theta = \tan\theta$$

$$\alpha z_R^{\ 3} + \beta z_R^{\ 2} + \gamma z_R + \delta = 0$$ *[Ref. Equation 32]*

$$(1)\tan^3\theta + \beta\tan^2\theta + \gamma\tan\theta + \delta = 0$$ *(for $\alpha=1$)*

Such that,

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

$$\zeta(1 - \gamma) + \beta = \delta$$

$$\zeta(1 - \frac{\beta^2}{3}) + \beta = \delta$$ *(for $\gamma = \beta^2/3$)*

Substitution into what appears below gives:

$$\tan^3\theta + \beta\tan^2\theta + \gamma\tan\theta + \delta = 0$$

$$\tan^3\theta + \beta\tan^2\theta + (\frac{\beta^2}{3})\tan\theta + [\zeta(1 - \frac{\beta^2}{3}) + \beta] = 0$$

$$\frac{\beta^2}{3}(\tan\theta - \zeta) + \beta(1 + \tan^2\theta) + \tan^3\theta + \zeta = 0$$

$$\beta^2 + [\frac{3(1 + \tan^2\theta)}{(\tan\theta - \zeta)}]\beta + (3)\frac{\tan^3\theta + \zeta}{(\tan\theta - \zeta)} = 0$$

Completing the square gives:

$$\beta^2 + [\frac{3(1+\tan^2\theta)}{(\tan\theta-\zeta)}]\beta + [\frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2 + (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)} = \frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2}$$

$$[\beta + \frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2 + (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)}(\frac{4}{4})(\frac{\tan\theta-\zeta}{\tan\theta-\zeta}) = \frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2}$$

Whereby,

$$[\beta + \frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)}]^2 = \frac{9(1+2\tan^2\theta+\tan^4\theta)}{4(\tan\theta-\zeta)^2} - (3)\frac{\tan^3\theta+\zeta}{(\tan\theta-\zeta)}(\frac{4}{4})(\frac{\tan\theta-\zeta}{\tan\theta-\zeta})$$

$$\beta + \frac{3(1+\tan^2\theta)}{2(\tan\theta-\zeta)} = [\frac{1}{2(\tan\theta-\zeta)}]\sqrt{9(1+2\tan^2\theta+\tan^4\theta)-12(\tan^3\theta+\zeta)(\tan\theta-\zeta)}$$

$$\beta = [\frac{1}{2(\tan\theta-\zeta)}][-3(1+\tan^2\theta) \pm \sqrt{9(1+2\tan^2\theta+\tan^4\theta)-12(\tan^4\theta-\zeta\tan^3\theta+\zeta\tan\theta-\zeta^2}]$$

Equation 52. ***Trisector Equation Generator for z_R=-β/3.***

$$\beta = [\frac{1}{2(\tan\theta-\zeta)}][-3(1+\tan^2\theta) \pm \sqrt{9+12\zeta^2-12\zeta\tan\theta+18\tan^2\theta+12\zeta\tan^3\theta-3\tan^4\theta}]$$

Therefore, for any *postulated real value* of $\zeta = \tan(3\theta)$ and its *associated, calculated value* $z_R = R\tan\theta = (1)\tan\theta = \tan\theta$, the *coefficients* β, $\gamma = \beta^2/3$, and $\delta = \zeta(1-\frac{\beta^2}{3})+\beta$ can be calculated in order to describe a *Generalized Cubic Equation* whose *root* is $z_R = -\beta/3$.

For the case when:

$$\zeta = \tan(3\theta) = 13/9$$
$$3\theta = 55.30484647^o$$
$$\theta = 18.43494882^o$$
$$z_R = \tan\theta = 1/3$$

Then, by applying *Equation 52*:

$$\beta = [\frac{1}{2(\tan\theta-\zeta)}][-3(1+\tan^2\theta) \pm \sqrt{9+12\zeta^2-12\zeta\tan\theta+18\tan^2\theta+12\zeta\tan^3\theta-3\tan^4\theta}]$$

$$= [\frac{1}{2(-10/9)}][-3(10/9) \pm \sqrt{(729+2028)/81-468/81+162/81+52/81-3/81}]$$

$$= [\frac{-9}{20}][-\frac{30}{9} \pm \frac{1}{9}\sqrt{(729+2028)-468+162+52-3}]$$

$$= [\frac{-1}{20}][-30 \pm \sqrt{2757-468+162+49}]$$

$$= \frac{3}{2} \mp \frac{1}{20}\sqrt{2500}$$

$$= \frac{3\mp 5}{2}$$

$$= -1;+4$$

$$\gamma = \frac{\beta^2}{3}$$
$$= \frac{1}{3}; \frac{16}{3}$$

$$\delta = \zeta(1-\gamma) + \beta$$
$$= \frac{13}{9}(1-\frac{1}{3}) - 1; \frac{13}{9}(1-\frac{16}{3}) + 4$$
$$= \frac{13}{9}(\frac{2}{3}) - \frac{27}{27}; \frac{13}{9}(-\frac{13}{3}) + 4(\frac{27}{27})$$
$$= \frac{1}{27}; -\frac{61}{27}$$

Hence, such above determined *coefficients* generate the following pair of *Generalized Cubic Equations*:

$$\alpha z^3 + \beta z^2 + \gamma z + \delta = 0$$ *[Ref. Equation 32]*

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$
$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

Check,

For

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma}$$ *[Ref. Equation 51]*

$$= \frac{3(\frac{13}{9})[\frac{1}{3} - 1(\frac{3}{3})] - 4(-1)(\frac{9}{9})}{3(\frac{9}{9}) + \frac{1}{3}(\frac{3}{3})}$$
$$= \frac{(\frac{13}{3})(\frac{-2}{3}) + (\frac{36}{9})}{\frac{27}{9} + \frac{3}{9}}$$
$$= \frac{10}{30}$$
$$= \frac{1}{3}$$

For

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$z_R = \frac{3\zeta(\gamma-1) - 4\beta}{3+\gamma}$$

$$= \frac{3(\frac{13}{9})[\frac{16}{3} - 1(\frac{3}{3})] - 4(4)(\frac{9}{9})}{3(\frac{9}{9}) + \frac{16}{3}(\frac{3}{3})}$$
$$= \frac{(\frac{13}{3})(\frac{13}{3}) - 4(4)(\frac{9}{9})}{\frac{27+48}{9}}$$
$$= \frac{25}{75}$$
$$= \frac{1}{3}$$

Also:

$$\beta^2 = 3\alpha\gamma \qquad\qquad \beta^2 = 3\alpha\gamma$$
$$= 3(1)(1/3) \qquad\qquad = 3(1)(16/3)$$
$$\beta = \pm\sqrt{1} \qquad\qquad \beta = \pm\sqrt{16}$$
$$\beta_2 = -1 \qquad\qquad \beta_1 = +4$$

So,

$$z_R = \frac{-\beta_2 + \sqrt[3]{\beta_2^{\,3} - 27\alpha^2\delta}}{3\alpha} = \frac{+1 + \sqrt[3]{(-1)^3 - 27(+1)^2(-\frac{1}{27})}}{3} = \frac{+1+\sqrt[3]{-1+1}}{3} = \frac{1+0}{3} = \frac{1}{3}$$

$$z_R = \frac{-\beta_1 + \sqrt[3]{\beta_1^{\,3} - 27\alpha^2\delta}}{3\alpha} = \frac{-4 + \sqrt[3]{(4)^3 - 27(1)^2(-\frac{61}{27})}}{3} = \frac{-4+\sqrt[3]{125}}{3} = \frac{-4+5}{3} = \frac{1}{3}$$

$$z^3 - z^2 + (1/3)z - 1/27 = 0$$
$$(\frac{1}{3})^3 - (\frac{1}{3})^2(\frac{3}{3}) + \frac{1}{3}(\frac{1}{3})(\frac{3}{3}) - \frac{1}{27} = 0$$
$$(1-3+3-1)/27 = 0$$
$$0 = 0$$

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$
$$(\frac{1}{3})^3 + 4(\frac{1}{3})^2(\frac{3}{3}) + \frac{16}{3}(\frac{1}{3})(\frac{3}{3}) - \frac{61}{27} = 0$$
$$\frac{1+12+48-61}{27} = 0$$
$$0 = 0$$

Now with regard to these newly determined equations, The ***common root*** $z_R = 1/3$ for the first *given Cubic Equation* above can be *geometrically constructed* without having to take a *cube root* since such ***cube root term*** *adds out to zero.*

Moreover, such first *given Cubic Equation*, as cited above, contains $z_R = 1/3 = -\beta/3$ as a root; thereby represents the tangent of the *trisected angle θ*, the latter of which then could be *geometrically constructed* very easily.

With regards to the *second above given Cubic Equation*, $z^3 + 4z^2 + (16/3)z - 61/27 = 0$, its *associated root* z_R can *be geometrically constructed* from its *given coefficients* via application of *Equation 51*, as shown above. Hence, in this particular case, it also is not necessary to obtain a ***cube root*** via *geometric construction*.

For such two *given Cubic* Equations, as are represented above, the following proof is provided in order to demonstrate that each relate to the same *angle* $3\theta = 55.30484647^o$:

For

$$z^3 - z^2 + \frac{1}{3}z - \frac{1}{27} = 0$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma} \text{ [Ref. Equation 36]}$$

$$= \frac{-\frac{1}{27} - (-1)(\frac{27}{27})}{1(\frac{27}{27}) - \frac{1}{3}(\frac{9}{9})}$$

$$= \frac{-1 + 27}{27 - 9}$$

$$= 26/18$$

$$\tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

For

$$z^3 + 4z^2 + \frac{16}{3}z - \frac{61}{27} = 0$$

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$

$$= \frac{-\frac{61}{27} - (4)(\frac{27}{27})}{1(\frac{27}{27}) - \frac{16}{3}(\frac{9}{9})}$$

$$= \frac{-61 - 108}{27 - 144}$$

$$= -169/-117$$

$$\tan(3\theta) = 13/9$$

$$3\theta = 55.30484647^o$$

Therefore, a *given angle* of $3\theta = 55.30484647^o$ can be ***divided*** *into three equal angles* of $\theta = 55.30484647^o / 3 = 18.43494882^o$ each by means of a *geometric construction* which utilizes nothing more than a *straightedge* and *compass* when ***applying*** the *coefficients* and *respective formats* expressed in *either* of the above determined *Cubic Equations*.

In conclusion, ***Generalized Cubic Equation formats*** exhibiting a ***sub-element*** *of* $R = 1$ contain a *root* of $z_R = \tan\theta$ with respect to their characteristic values of $\zeta = \tan(3\theta)$ such that,

$$\zeta = \frac{\delta - \beta}{1 - \gamma}$$ *[Ref. Equation 36]*

Such values z_R and ζ can be determined by a *geometric construction* which employs *only straightedge and compass* instruments that operate solely upon *various inherent coefficients* resident within these *formats*.

Since an ***angle of 3θ*** can be ***geometrically constructed*** from a *given* value of $\zeta = \tan(3\theta)$, and since an ***angle of θ*** also can be ***geometrically constructed*** from such previously *algebraically determined* value of $z_R = \tan\theta$, trisection can be *achieved* through *geometric manipulation* of such *inherent coefficients*.

This does not constitute a bonafide *Euclidean Trisection* event, however, since such ***Generalized Cubic Equation formats*** exist merely as algebraic *transformations* that constitute ***aforehand knowledge*** of such *desirable root structures* in the first place *(Ref. Section 19)*.

SECTION 22. *PORTRAYING LENGTHS OF CUBIC IRRATIONAL VALUES FROM UNITY.*

(In the event of any conflict between this section and U.S. Patent No. 10994569 issued on 5/4/2021, the latter governs)

Astounding as it might sound, *cubic irrational lengths* actually *can* be **portrayed** from any *arbitrarily assigned* or *given* length of unity, while still adhering to all of the precepts espoused in the *conclusion* given in *Section 9.1*.

This is to be achieved by a process whereby *cubic irrational lengths* become ***geometrically formed*** instead of *geometrically constructed*.

Furthermore, *trisected angles*, respectively equal to exactly one-third the magnitude of any given angles, now also shall be capable of being portrayed in this manner!

With respect to the above, the prospect of identifying *cubic irrational lengths* is considered to be of ***far greater importance*** than *trisecting* various given angles of 3θ; essentially because the concept of depicting *exact cubic irrational lengths* alongside an amalgamation of *rationally-based lengths* that actually define them should exemplify a fitting or fundamentally new ***Number Theory*** groundwork; one from which amazing, new discovery may be launched, and one which should serve to appreciably *advance* the overall state-of-the-art!

In contrast, an ability to *trisect an angle*, although of *significant import*, nevertheless does not exemplify this same *profound capability* to stand alone as an actual ***Number Theory*** groundwork in itself; one from which other meaningful applications could then become derived.

22.1. Geometrically Formed Lengths Whose Magnitudes are of Cubic Irrational Value.

Geometrically formed cubic irrational lengths become evident during ***overlapment***, a singular condition observed to occur whenever the *longitudinal axis* of a pre-selected *compass arm* (belonging to a new appurtenance consisting *solely* of *compass* and *straightedges* interconnected in a unique manner) hovers *directly over* the *determinable point* (η, τ).

Cubic irrational lengths result because ***geometric constraint*** becomes imposed upon the *endpoint* of the other compass arm.

Setting all *compass arm* and *straightedge* lengths equal to an arbitrary value of *unity* assures that resulting *cubic irrational lengths* can become depicted directly alongside such *rational unitary basis*.

22.2. The Euclidean Limitation.

The capability to ***geometrically construct*** circles about any given location serving as a center point became possible with the advent of the *compass.*

Coupled with an ability to connect two points via *straightedge*, the very ***groundwork*** for an entire *branch of mathematics* came into being.

Thereafter, it became evident that ***geometry***, unfortunately, appeared to be complete in every major respect, save one:

- It exhibited the *flaw* or limitation of being able to replicate *only* ***rationally-based lengths***!

Although entirely ***accurate*** in its *precepts* and *geometric renderings*, however, by lacking an innate capability to depict *cubic irrational lengths*, such *Euclidean practice* today looms as being somewhat ***incomplete***!

Euclidean practice is governed by ***geometric construction***. It remains severely hampered simply because it has not been permitted to be analyzed beyond such *confining borders*!

To elaborate, *Euclidean practice* first should be defined in a manner conducive to everybody's liking. Naturally, that would consist of describing it with respect to itself!

Such statement might well assert, "*Euclidean practice* is exactly what it is stacked up to be; that is, exactly what it has been considered to accomplish since its very inception – No more, and no less"!

Stemming from such logic, another leading question is whether *Euclidean practice* is ***all*** that it *needs* to be.

In order to answer, a *reasonable* ***basis*** for *Euclidean practice* would have to be established; one considered being seemingly acceptable to the entire mathematics community.

Such plausible basis unequivocally would hereinafter state,

- "*Euclidean renderings* should be what *can* be portrayed via *compass* and *straightedge* tools starting only from an arbitrarily assigned, or given length of unity -- No more, and no less" *(Ref. Section 9.1 Conclusion)*!

Accordingly, what appears to be at issue here is, not the *adequacy* of *Euclidean practice*, but its very ***relevancy***.

22.3. The Present Day Euclidean Realm.

With regard to the rules and regulations which apply to *Euclidean practice*:

22.3.1. The Introduction of Relative Motion.

The *compass* may be perceived as an instrument which applies ***relative motion*** in itself, as evidenced by the process of swinging a *circular arc*.

During such process:

- One endpoint, or tip of the *compass* is permitted to rotate, but not to displace away from a *stationary center point*
- The other endpoint of the compass, during such rotation, is free to ***geometrically construct*** a circumference, thereby depicting a locus of points that becomes set a fixed distance away from such *center point*

Such compass consists of two arms which remain fixed, or maintained stationary with respect to one another via tightened hinge. As an integral unit, it becomes free to rotate about the compass tip located directly above this center point, and thereafter to *relatively describe* such aforementioned circumferential loci.

Relative motion actually manifests itself as the *relative movement* between such ***stationary*** *center point* and the displaced *compass tip* during conditions when it describes a circle.

Notice that the compass can operate *properly* only when ***overlapment*** becomes imposed; thereby maintaining the endpoint of its non-displaced arm directly over such *stationary center point* at all times during rotation.

Such motion either may be *prematurely terminated* when the circumference being described encounters another independent *geometric construction*; or may be perpetuated until a full circle becomes embellished, whereby such intersection point(s) become identified afterwards.

Secondly, it may have been *Archimedes* (287 – 227 B.C) who first was cited for attempting to apply <u>additional</u> ***relative motion*** during his attempts to achieve *trisection*.

In so doing, such **Neusis construction**[1,2,3] identifies a *given angle* 3θ that is to be *trisected* with its vertex residing at a *Point O*. A *singular straight line* then is constructed *parallel* to *its base*. The intersection point of such *parallel line* with the other side of such *angle* eliciting trisection is to be designated as *Point A*.

Length $\overline{OA}$ then is arbitrarily designated as one unit of measurement. Thereafter, a *circle* is to be ***geometrically constructed*** of radius one unit whose *origin* resides at *Point A*. Naturally, its circumference passes through *Point O*.

Completely independent of such *construction*, a *straightedge* then is configured such that it contains *two notches* upon it set a distance *apart* of *unity*.

The <u>non-Euclidean</u> aspect of *Archimedes'* approach is that to achieve trisection, the *straightedge* then becomes ***moved,*** in a completely *arbitrary*, or *haphazard fashion* until it passes through *Point O*, whereby one of its *notches* becomes aligned upon the *circumference* of such predetermined *circle* and its other somewhere upon the *parallel straight line*.

22.3.2. The Imposition of Geometric Constraint.

Geometric constraint is <u>not</u> new to *Euclidean practice* either. This is evidenced by the *tightening mechanism* on a *compass* which limits *circumferences* of only a *single* specified radius to become ***geometrically constructed***.

1. *Internet website*: http://en.wikipedia.org/wiki/Angle_trisection#cite_note-9.
2. <u>A Budget of Trisections</u>, U. Dudley; Springer-Verlag; 1987; pages 3-4.
3. <u>What is Mathematics</u>, H. Robbins and R. Courant; Oxford University Press; October 1947; page 138.

22.3.3. Overlapment.

In particular, ***overlapment*** pinpoints a *singular location* along the *longitudinal axis* of a straightedge which just so happens to *superimpose upon*, or *coincide with* a *determinable point* in space; one which either may be stationary, or moving itself.

From the distant vantage point of Earth, such distinct *longitudinal axis*, once contemplated to exist outside of the realm of such aforementioned appurtenance *(Ref. Section 22.1)*, may be perceived as a straight line of seemingly imperceptible width which becomes drawn, for example, through *Orien's Belt*. At the precise moment when it is observed to pass either directly in front of or behind a particular star, no matter how faint, ***overlapment*** occurs at the specific location where such straight line is viewed to cross, or intersect *with* the star.

Such process also may be likened to a *total eclipse* of the sun by the moon. During this occurrence, a straight line fictitiously can be drawn which is considered to *intersect*:

- The center of the moon
- The center of the sun
- The midway point between the viewer's eyes

Overlapment coexists with *intersection*. They go hand-in-hand, whereby at times they even might be perceived as being *inextricably linked* or associated to one another.

The only ***difference*** between them is that ***overlapment*** seeks to identify additional *intersection points* that previously either went undetected or otherwise were deliberately ignored during prior *geometric construction* exercises.

Had *Euclid* and his contemporaries been advised that *cubic irrational lengths* actually could be depicted solely from a unique arrangement of *compasses* interconnected via *straightedge*, such capability most definitely would have been incorporated into their practice long ago.

Such esoteric notion of intersection points, considered to be germane to ***geometry***, nevertheless remains *fundamental* to generally accepted *Euclidean practice.* As such, the newly fashioned property of ***overlapment***, because it also locates such points, should be categorized under this very same, overall *Euclidean umbrella*.

22.4. The Proposed Euclidean Enhancement.

The prospect of incorporating ***cubic irrational length*** *depictions* into formerly established *Euclidean practice* without violating, detracting from, or otherwise conflicting with its precepts *theoretically* would entail:

- Using only *Euclidean compass* and *straightedge* instruments in a manner entirely consistent with all of the rules and regulations applied during Euclid's day
- Treating ***cubic irrational length geometrically formed*** *depiction* in exactly the same manner as ***rationally-based geometric construction***; whereby both become determinable entirely from a given length of unity *(Ref. Section 9.1)*
- Acknowledging the *process* of obtaining ***geometrically formed*** depictions as a new Euclidean ***enhancement***; one which remains completely independent, or is distinguished entirely apart from the presently accepted *Euclidean process* of ***geometric construction***; one conceived perhaps after the metaphor:
  - As the *process* of ***geometric construction*** becomes likened to the branch of *physics* known as *Statics*
  - The process of obtaining ***geometrically formed*** depictions then might be likened to its other branch, known as *Dynamics*

Were the ***geometric construction*** *practice* of identifying *intersection points* to become *supplemented* by including those which also become identified during ***overlapment***, then a complete capability to portray both *rationally-based* and *cubic irrational lengths* would result!

Conversely, that entire *branch of mathematics*, otherwise known as ***geometry***, will never be complete in all respects until it takes up the ball by considering the extraordinary consequences of ***overlapment***!

Because of this lingering omission, *it is recommended* that such *enhancement* be incorporated into *Euclidean practice* via amendment in order to finally avail a new capability for actually depicting exact *cubic irrational lengths* ***geometrically***!

22.5. Theory.

Cubic irrational numbers are known to manifest themselves as *cubic root values* z_R, z_S, and z_T inherent within ***3θ Cubic Equation****s*.

In consonance with the *Cubic Equation Cubic irrational Root Uniqueness Theorem*, reiterated below, this may be interpreted to mean *(Ref. Section 9.3)*:

> *"Only Cubic Equations allow solely rationally-based numerical coefficients to co-exist with root sets comprised of trigonometric, cubic irrational numbers".*

When a ***3θ Cubic Equation***, of the particular form designated below, possesses a *rationally-based coefficient* of $\zeta = \tan(3\theta)$, its *roots* nevertheless still may be *cubic irrational*.

$$z^3 - 3\zeta z^2 - 3z + \zeta = 0$$

During such circumstances, a ***mutual existence*** between equation *rationally-based coefficients* and *associated cubic irrational roots* presumably occurs.

As such, emphasis hereafter is to be placed upon $\zeta = \tan(3\theta)$ values which specifically consist either of:

a) *Rationally-based lengths (Ref. Section 9.1)*; or
b) *Cubic irrational root lengths* ascertained from them.

Now, *Table 35* gives a first hand accounting of z_R *cubic irrational root length values* which were *ascertained from* two sample *3θ Cubic Equations* which respectively exhibit the following $\zeta = \tan(3\theta)$ rationally-based values:

1) $\zeta = \sqrt{3}$; and

2) $\zeta = (3/8)\sqrt{57}$.

Table 35 relates how *cubic irrational root length values* ascertained from such specific *rationally-based values* become commissioned as actual ζ values in themselves, in order to perpetuate *numerical length determinations*.

In *Table 35:*

- *R* stands for *Rational Length Classifications*
- *R-B* stands for *Rationally-based Length Classifications*
- *C* stands for *Cubic irrational Length Classifications*

Table 35. Examples of Length Lineages whose Magnitudes are of Cubic Irrational Value.

START		ζ=TAN 3θ		3θ	θ	Z_R=TAN θ	
VALUE	CLASS.	VALUE	CLASS.	VALUE (Deg.)	VALUE (Deg.)	VALUE	CLASS.
UNITY	R	$\sqrt{3}$ =1.732050808	R-B	60	20.00000000000000	0.363970234266202	C
WHEN UNITY BEGETS A *RATIONALLY-BASED* LENGTH OF $\zeta = \sqrt{3}$		0.363970234266202	C	20	6.66666666666667	0.116883236758153	C
		0.116883236758153	C	6.66666666666666	2.22222222222222	0.038804554124571	C
		0.038804554124571	C	2.22222222222222	0.74074074074074	0.012929085171325	C
		0.012929085171325	C	0.74074074074074	0.24691358024691	0.004309481620934	C
		0.004309481620934	C	0.24691358024691	0.08230452674897	0.001436485969123	C
		0.001436485969123	C	0.08230452674897	0.02743484224966	0.000478828363616	C
		0.000478828363616	C	0.02743484224966	0.00914494741655	0.000159609443696	C
		0.000159609443696	C	0.00914494741655	0.00304831580552	0.000053203147497	C
		0.000053203147497	C	0.00304831580552	0.00101610526851	0.000017734382484	C
		0.000017734382484	C	0.00101610526851	0.00033870175617	0.000005911460827	C
		0.000005911460827	C	0.00033870175617	0.00011290058539	0.000001970486942	C
		0.000001970486942	C	0.00011290058539	0.00003763352846	0.000000656828981	C
		0.000000656828981	C	0.00003763352846	0.00001254450949	0.000000218942994	C

START		ζ=TAN 3θ		3θ	θ	Z_R=TAN θ	
VALUE	CLASS.	VALUE	CLASS.	VALUE (Deg.)	VALUE (Deg.)	VALUE	CLASS.
UNITY	R	$(3/8)\sqrt{57}$ =2.831187913	R-B	70.54633985	23.51544662	0.435132976886788	C
WHEN UNITY BEGETS A *RATIONALLY-BASED* LENGTH OF $\zeta = (3/8)\sqrt{57}$		0.435132976886788	C	23.51544662	7.83848220602824	0.137667267797039	C
		0.137667267797039	C	7.83848220602824	2.61282740200941	0.045634078616798	C
		0.045634078616798	C	2.61282740200941	0.87094246733647	0.015201984549349	C
		0.015201984549349	C	0.87094246733647	0.29031415577882	0.005066981246551	C
		0.005066981246551	C	0.29031415577882	0.09677138525961	0.001688980900521	C
		0.001688980900521	C	0.09677138525961	0.03225712841987	0.000562993157648	C
		0.000562993157648	C	0.03225712841987	0.01075237613996	0.000187664368258	C
		0.000187664368258	C	0.01075237613996	0.00358412537999	0.000062554788767	C
		0.000062554788767	C	0.00358412537999	0.00119470846000	0.000020851596231	C
		0.000020851596231	C	0.00119470846000	0.00039823615333	0.000006950532076	C
		0.000006950532076	C	0.00039823615333	0.00013274538444	0.000002316844025	C
		0.000002316844025	C	0.00013274538444	0.00004424846148	0.000000772281342	C
		0.000000772281342	C	0.00004424846148	0.00001474948716	0.000000257427114	C

START		ζ=TAN 3θ		3θ	9θ	Z_R=TAN (9θ)	
VALUE	CLASS.	VALUE	CLASS.	VALUE (Deg.)	VALUE (Deg.)	VALUE	CLASS.
UNITY	R	$(3/8)\sqrt{57}$ =2.831187913	R-B	70.54633985	211.6390196	0.616143267895401	C
		0.616143267895401	C	31.63901956	94.91705869	-11.623829453592200	C
		-11.623829453592200	C	-85.08294131171300	-255.2488239	-3.797949431449620	C
WHEN UNITY		-3.797949431449620	C	-75.24882393513900	-225.7464718	-1.026402254289320	C
		-1.026402254289320	C	-45.74647180541710	-137.2394154	0.924733142374527	C
BEGETS AN		0.924733142374527	C	42.76058458374870	128.2817538	-1.267048998398790	C
R-B		-1.267048998398790	C	-51.71824624875400	-155.1547387	0.463023837085188	C
		0.463023837085188	C	24.84526125373800	74.53578376	3.614648367324490	C
OF		3.614648367324490	C	74.53578376121390	223.6073513	0.952531816927358	C
		0.952531816927358	C	43.60735128364170	130.8220539	-1.157610030597300	C
$\zeta = \frac{3}{8}\sqrt{57}$		-1.157610030597300	C	-49.17794614907500	-147.5338384	0.636240283748642	C
		0.636240283748642	C	32.46616155277510	97.39848466	-7.701168183402210	C
		-7.701168183402210	C	-82.60151534167470	-247.804546	-2.450981071377900	C
		-2.450981071377900	C	-67.80454602502400	-203.4136381	-0.433021274698496	C

The top two charts expressed in *Table 35* portray z_R values as tan θ quantities, thereby associating them with *trisected angles* of 3θ.

The bottom chart portrays z_R values as tan (9θ) quantities, thereby associating them with angles which are equal in magnitude to three times that of accorded 3θ angles.

Quite obviously, other *trigonometric depictions* besides those specified in *Table 35* can be determined as *offshoots* to such tangent determinations -- including both sine and cosine portrayals.

Even though ***all*** cubic irrational lengths (such as the value for $\pi = 3.14159265358979323846264338327 9\ldots$) quite possibly cannot yet directly be ascertained via this above process, nevertheless it still significantly and sufficiently contributes to the ***overall advancement of Number Theory***, simply because it now equips humanity with a brand new, profound capability to actually depict *cubic irrational numbers* ***geometrically*** *(Ref. Related Problem Number 48)*!

22.6. The Advent of Atacins.

Hereinafter, only the *compass* and *straightedge* from bygone days, along with a *given* or *arbitrarily assigned length of unity* need to be applied in future endeavors.

Perhaps it has been mankind's *mathematical destiny*, one possibly attributed to his obstinacy, complicit acceptance of, *total adherence to*, or even his very preoccupation, infatuation, or obsession with such prior *Euclidean practice* which explains exactly why a process to obtain ***geometrically formed*** *cubic irrational lengths* never was established until now, even though *twenty-three hundred years* has transpired since such discoveries first occurred!

And so, at last it finally is recommended that a great leap should be made to apply ***relative motion*** along with its ***constituent constraint*** in a novel way; one which so sorely has been lacking in prior *Euclidean practice*.

The ***Atacins*** is a new device which creates a ***geometrically formed*** depiction of an angle exactly one-third the magnitude of any *given angle* that is programmed into it. Even when the tangent of such *resulting angle* is a *cubic irrational length*, **Atacins** exactly depicts it. Its name is an *acronym* for ***a****ngle* ***t****risector* ***a****nd* ***c****ubic* *irrational length* ***i****nstrument*, whereby a motion must be *imparted* during such determinations.

The device *overcomes* the *rational number* to *cubic irrational number quandary* normally experienced during prior attempts at *Euclidean trisection*. This is achieved by articulating such invention until its *two compasses* ***overlap*** one another in a prescribed fashion; whereby, *cubic irrational length*s become ***depicted*** alongside *given rationally-based* ones.

The *Atacins* adapts to angles of any *position* or *orientation*, thereby overcoming any inadequacy which might be associated with the *tomahawk*[1].

Now, the *compass* may be defined as a *functioning mechanical device* whereby *straightedges* of *equal length* are connected by a *hinge*. As this *hinge* becomes *tightened*, a prescribed, *unvarying radius or* ***hinge angle*** becomes *set* which allows a *circle* of *particular arc* to be *geometrically constructed*.

1. Section entitled, *With a "Tomahawk"* appearing in *Internet website*: http://en.wikipedia.org/wiki/Angle_trisection#cite_note-9.

Alternatively, such *prescribed radius* or *fixed circular arc*, once defined, may be maintained by affixing a *third member* between the *open ended* ***terminations*** of such two *compass straightedge arms*, thereby comprising an *isosceles triangle* which replaces, or obviates the need for such *tightening mechanism*.

The *Atacins* consists of two such *isosceles triangle arrangements* each of whose respective hinges (see above) are set apart by and interconnected to opposite *termination points* of a *third straightedge* of *length* equal to that of respective *isosceles triangle equal length arms*.

In essence then, *two hinges* attach *middle straightedge* endpoints to respective *assemblies* of *swinging arms* which collectively may be actuated as *independent compasses*.

Furthermore, *third members* of each *swinging arm assembly* are retained by *secondary hinges* at one end and fitted with *slots* that house screw and nut assemblies on the other in order to enable such aforementioned *prescribed radius* or *fixed circular arc* to be described and then set.

The fact that *dual compasses* are afforded, set apart from one another by a known or established distance, is completely *consistent* with *Euclidean* ***geometric construction*** techniques since this is equivalent to taking *a single compass* and swinging a circular arc of *known radius* about one fixed point, and then moving the compass to another prescribed location and swinging a second, independent circular arc of equal radius.

The *Atacins* imposes *Euclidean geometric constraint* via:

- Two *third member slotted arms* which are equipped with respective retained screw and nut arrangements. These features enable a specific radius or *common* ***hinge angle*** to be set, or programmed in.

  [Bear in mind that such Atacins tightening mechanisms ***alternately*** *can assume the same exact design configuration, or features, evident within present day compasses, thereby affording virtually identical Euclidean geometric constraint.]*

- Additionally, one other *strategically emplaced slot*, etched into another *compass arm*, as elaborated upon below, imposes yet a third *Atacins* ***constraint***.

22.6.1. Device Embodiment.

Specifically, the *Atacins* is a mechanism consisting of a *middle straightedge member* which *interconnects* with two *independent assemblies* comprised of *identically shaped isosceles triangles*. Moreover, the assigned length of such *middle straightedge* is equal to that applied to each equal side of both isosceles triangles.

For each *isosceles triangle*, *enclosed angles* located adjacent to such *middle straightedge member* are to be adjusted to a *known* or *given angle* of $(90-3\theta)^{\circ}$. This is achieved by precisely locating each *third member slot arrangement* with respect to the free-moving arm that it captures. Once set to such angle, however, each of the isosceles triangle members are to become *fixed*, or rigid, for the remainder of each ongoing construction by tightening respective screw and nut assemblies.

Figure 50 illustrates that such *adjustment* may be accomplished merely by means of *slotting* the *unequal side* of each *isosceles triangle*; whereby hinges, or rivets, are inserted to secure two of its three vertices. To allow for easy rotation, each rivet should be a little deeper than the combined width of contained *Atacins* members that it retains. The remaining vertex of each *isosceles triangle* is to incorporate a screw and nut arrangement that sits, or is permitted to ride within such slot.

The two isosceles triangles along with its straightedge member may be comprised of virtually any inexpensive material.

Figure 50. Atacins Isosceles Triangular Members.

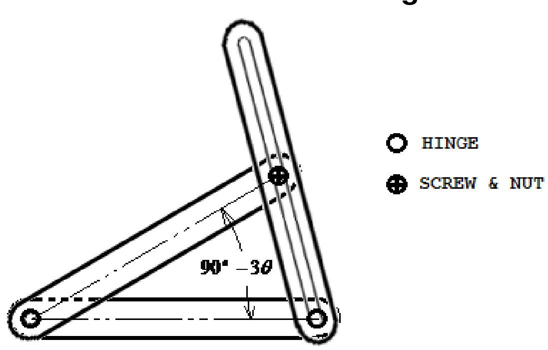
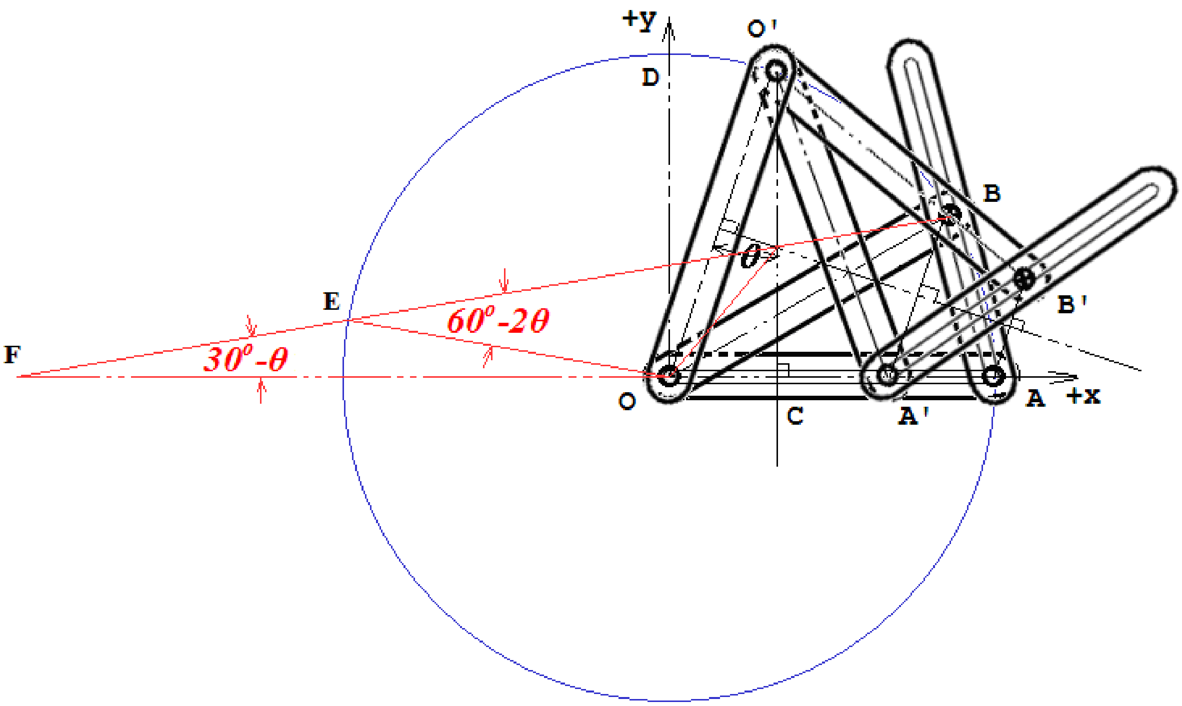


Figure 51 shows the *Atacins* configuration. It consists of *middle straightedge member* $\overline{OO'}$, along with two *isosceles triangular members* denoted as OAB and O’A’B’, respectively. Both *triangular members* are free to pivot about connection hinges placed at points O and O’, respectively.

Figure 51. Atacins Configuration (Now Patented).



Member $\overline{OA}$ also is to become *slotted* for purposes of capturing the A′ vertex of *isosceles triangle* O′A′B′. This serves to thereby ***constrain*** the *relative movement* of such vertex to just *horizontal displacements* located anywhere along such slot.

The *Atacins* is operated in a single motion by rotating member $\overline{OO'}$ with respect to member $\overline{OA}$ about its *retained hinge* located at point O. As member $\overline{OO'}$ moves in the clockwise direction, it engages member $\overline{O'A'}$ whose termination at point A′, in turn, displaces to the right. This forces isosceles triangle O′A′B′ to rotate counterclockwise. At such point when arm $\overline{O'B'}$ ***overlaps***, or *hovers* directly over point B, ∠OO′C becomes <u>exactly</u> equal to θ. Such arrangement permits the actual *geometric co-existence* between a *given angle* $(90-3\theta)$ and an *angular depiction* of *θ*. In other words, it ***geometrically resolves*** the Euclidean *straightedge* and *compass* quandary of formerly attempting to associate solely *rational coefficients* expressed in a given *Cubic Equation* with its respective *cubic irrational roots (Ref. Section 21)*.

For ease of construction, all seven members of the *Atacins* are to be of the *same exact length*, say unity.

This includes:

- $\overline{OA}$
- $\overline{OB}$
- $\overline{OO'}$
- $\overline{O'A'}$
- $\overline{O'B'}$
- $\overline{AB}$ *extended*
- $\overline{A'B'}$ *extended*

In total, all members may be virtually identical, with the only exception being that members $\overline{OA}$, $\overline{AB}$ and $\overline{A'B'}$ are to be *slotted*.

22.6.2. Proof.

The Atacins proof, presented below, hinges upon the premise that when the device becomes set into motion, certain *mathematical relationships* are ***maintained***, regardless of what magnitude of $(90\text{-}3\theta)^{\circ}$ is applied. In other words, it validates that for each *discrete value* of $\angle AOB = \angle A'O'B' = (90-3\theta)$ that is afforded, *∠OO'C always* turns out to be exactly equal to *θ (Ref. Figure 51)*.

ACTION	PROOF
1. In *Figure 51*, draw circle with center at "O" and radius $\overline{OA}$.	1. Construction
2. Now draw radius $\overline{OO'}$ so that Point O' resides on circumference of circle.	2. Construction
3. $\angle O'OD = \theta$	3. Assignment
4. Draw an angle equal to $(90\text{-}3\theta)^{\circ}$ with the x-axis	4. Construction
5. Designate such $(90\text{-}3\theta)^{\circ}$ angle as $\angle$AOB with $\overline{OA}$ and $\overline{OB}$ being radii	5. Assignment
6. Draw straight line $\overline{O'C}$ parallel to the y-axis	6. Construction
7. $\angle OO'C = \angle O'OD = \theta$	7. Alternate interior angles are equal *(Ref. line 3)*
8. $\angle$AOD=90^{o}	8. Circle quadrant contains 90^{o} of arc
9. $\angle AOB + \angle BOD = \angle AOD$	9. The whole is equal to the sum of its parts
10. $90^{o} \text{ - } 3\theta + \angle BOD = 90^{o}$	10. Substitution of line 5 and line 8 into line 9
11. $\angle BOD = 3\theta$	11. Subtracting 90^{o}-3θ from each side of the equation
12. $\angle BOO' + \angle O'OD = \angle BOD$	12. The whole is equal to the sum of its parts
13. $\angle BOO' + \theta = 3\theta$	13. Substitution of line 3 and line 11 into line 12
14. $\angle BOO' = 2\theta$	14. Subtracting θ from each side of the equation
15. Draw chord $\overline{O'B}$	15. Construction
16. Triangle BOO' is isosceles	16. 2 sides of the triangle are comprised of radii of equal lengths
17. $\angle OBO' = \angle OO'B$	17. Angles opposite equal sides of an isosceles triangle are equal

ACTION	PROOF
18. $\angle OBO' + \angle OO'B + \angle BOO' = 180^o$	18. The sum of the internal angles of a triangle equal 180^o
19. $\angle OO'B + \angle OO'B + 2\theta = 180^o$	19. Substitution of line 14 and line 17 into line 18
20. $2(\angle OO'B) + 2\theta = 180^o$	20. Summing like terms
21. $\angle OO'B + \theta = 90^o$	21. Multiplying each side of the equation by ½
22. $\angle OO'B = \angle OO'B' = (90 - \theta)^o$	22. Construction & Subtracting θ from each side of the equation
23. Locate point A’ by swinging radius $\overline{O'O}$ until it intersects with the x-axis	23. Construction
24. Draw straight line $\overline{O'A'}$	24. Construction
25. $\overline{O'A'} = \overline{O'O}$	25. Radii of a circle are all of equal length
26. Triangle OO’A’ is isosceles	26. 2 sides of the triangle are of equal length
27. $\angle COO' + \angle O'OD = \angle AOD$	27. The whole is equal to the sum of its parts
28. $\angle COO' + \theta = 90^o$	28. Substitution of line 3 and line 8 into line 27
29. $\angle COO' = (90 - \theta)^o$	29. Subtracting θ from each side of the equation
30. $\angle CA'O' = \angle COO'$	30. Angles opposite equal sides of an isosceles triangle are equal
31. $\angle CA'O' = (90 - \theta)^o$	31. Substitution of line 29 into line 30
32. $\angle A'CO' = 90^o$	32. Altitude $\overline{O'C}$ divides isosceles triangle OO’A’ in 2
33. $\angle A'CO' + \angle CA'O' + \angle A'O'C = 180^o$	33. The sum of the internal angles of a triangle equal 180^o
34. $90^o + (90 - \theta)^o + \angle A'O'C = 180^o$	34. Substitution of line 31 and line 32 into line 33
35. $\angle A'O'C = \theta$	35. Adding $\theta - 180^o$ to each side of the equation
36. $\angle A'O'O = \angle A'O'C + \angle OO'C$	36. The whole is equal to the sum of its parts
37. $\angle A'O'O = \theta + \theta$	37. Substitution of line 7 and line 35 into line 36
38. $\angle A'O'O = 2\theta$	38. Summing like terms
39. $\angle A'O'O + \angle A'O'B' = \angle OO'B'$	39. The whole is equal to the sum of its parts
40. $2\theta + \angle A'O'B' = (90 - \theta)^o$	40. Substitution of line 22 and line 38 into line 39
41. $\angle A'O'B' = (90 - 3\theta)^o$	41. Subtracting 2θ from each side of the equation

22.6.3. Alternate Configurations.

22.6.3.1. Replacing Members $\overline{O'A'}$ and $\overline{A'B'}$ by Member $\overline{OB'}$.

The analysis below validates that existing *members* $\overline{O'A'}$ and $\overline{A'B'}$, along with the slot resident within *member* $\overline{OA}$, can be *replaced* by *member* $\overline{OB'}$, a new member of length greater than unity which is to be *slotted* along its entire length, except for where it interconnects with the *point O* hinge.

As illustrated in *Figure 51*, once $\angle AOB$ and ***to be replaced*** $\angle A'O'B'$ each are set equal $(90 - 3\theta)^o$, *member* $\overline{O'B'}$ has its:

1) End-point O’ residing somewhere along the circumference of a circle described about Point O of radius $\overline{OO'}$; and

2) End-point B′ residing upon the ***bisector*** of $\angle AOB$ because when *member* $\overline{O'B'}$ becomes located directly over point B:

$\angle OO'B' = (90-\theta)^o$ *[Ref. line 22 of Section 22.6.2 Proof]*

Now, since the *internal angles* within *isosceles triangle OO′B′* are equal to 180 degrees:

$$\angle OB'O' + \angle B'OO' + \angle OO'B' = 180^o$$

Furthermore, inherent angles opposite its respective equal sides must be equal as follows:

$$\angle OB'O = \angle B'OO'$$

Via substitution above,

$$\angle B'OO' + \angle B'OO' + \angle OO'B' = 180^o$$

$$\angle B'OO' + \angle B'OO' = 180^o - \angle OO'B'$$

$$\angle B'OO' = \frac{180^o - \angle OO'B'}{2}$$

$$= \frac{180^o - (90-\theta)^o}{2}$$

$$= \frac{(90+\theta)^o}{2}$$

Via *line 29* substitution *(Ref. Section 22.6.2 Proof)*, at:

$$\angle AOO' = \angle AOB' + \angle B'OO' = \angle COO'$$

$$\angle AOB' = \angle COO' - \angle B'OO'$$

$$= \frac{2}{2}(90-\theta)^o - \frac{(90+\theta)^o}{2}$$

$$= (90-3\theta)^o / 2$$

This above proof holds for all values of θ, thereby allowing *Atacins* to successfully trisect *any given angle* via ***overlapment*** of member $\overline{O'B'}$ and designated point B.

22.6.3.2. Parallelogram Construction.

Transforming the *Atacins* into a *parallelogram* hinged at all four corners assures that during flexure, or actuation, all respective opposite sides remain *parallel*. This is achieved by now replacing *members* $\overline{A'B'}$ and $\overline{O'B'}$ by two new *members* described as follows *(Ref. Figure 51)*:

1) A first *added member* of length $\overline{OO'}$ is to be retained on one end via the hinge located at *Point O′*, while its other free end is then positioned directly to the right, thereby rendering such new member parallel to member $\overline{OA}$. The *new member* also is to be *slotted* exactly in the same way as *member* $\overline{OA}$;

2) A second *added member*, also of length $\overline{OO'}$, is to be retained on one end via the hinge located at *Point A*, and positioned to runs parallel to member $\overline{OO'}$.

Where these two *new members* meet, they again are to become hinged at their respective unrestricted ends.

Lastly a *third added member*, also of length $\overline{OO'}$, is to be retained on one end via the hinge located at *Point A'*, with its other end captured within the slot arrangement of the *first new member* described above.

Due to the *constraints* imposed by the *parallelogram*, such *third member* then will re-position itself *parallel* to member $\overline{OO'}$ during all flexures or actuations of the device.

As such *third member* hovers directly over *Point B*, it also then will assume the same *orientation* with respect to the y-axis as member $\overline{OO'}$; an angle precisely equal to *θ* as verified below:

$$
\begin{aligned}
\sin(2\theta) &= \sin(3\theta-\theta)\\
&= \sin(3\theta)\cos\theta-\cos(3\theta)\sin\theta\\
2\sin\theta\cos\theta &= \eta\cos\theta-\tau\sin\theta\\
2\sin\theta &= \eta-\tau\tan\theta\\
\tan\theta &= \frac{\eta-2\sin\theta}{\tau}\\
&= \frac{\eta-\overline{OA'}}{\tau}
\end{aligned}
$$

Such above proportion holds whenever member $\overline{OO'}=\overline{OB}$ is set equal to one unit of measurement, thereby locating *Point B* at the following coordinates:

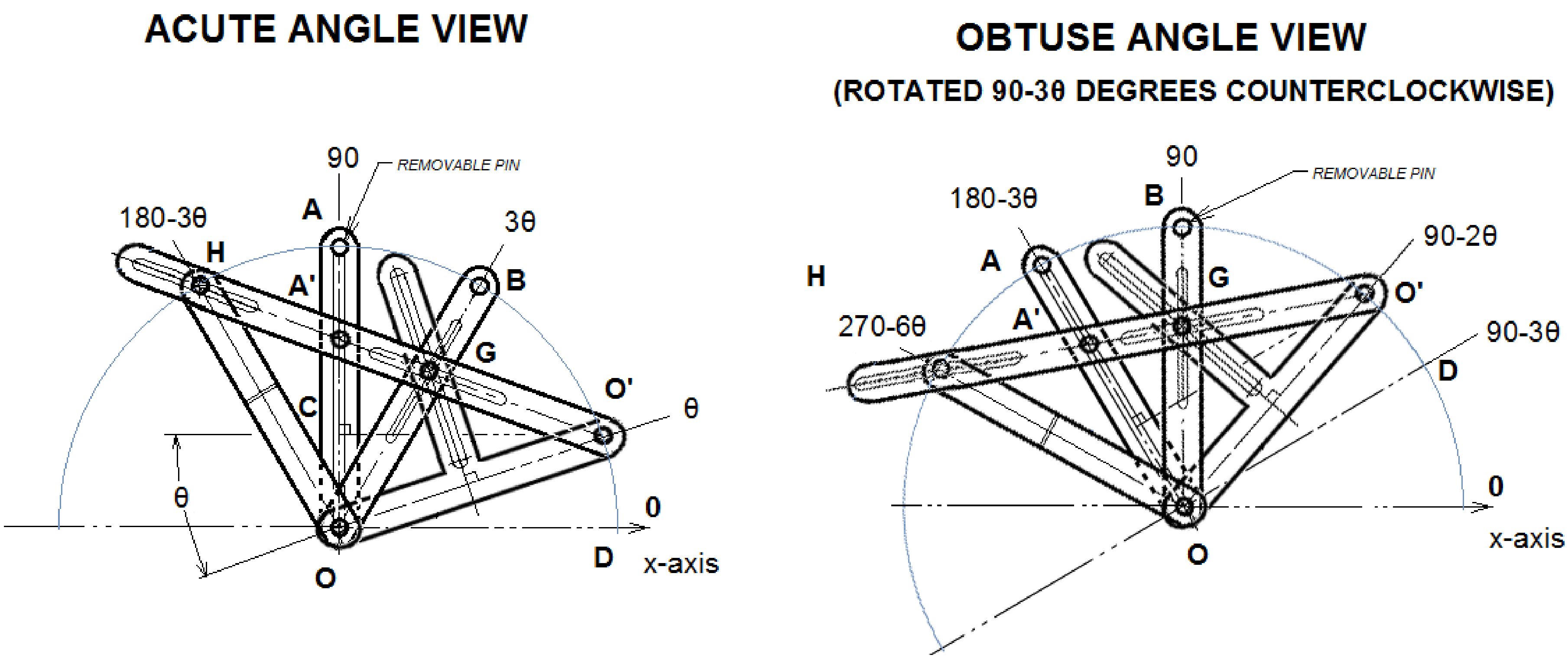
$$B_x;B_y=\eta;\tau$$

22.6.3.3. Car Jack Configuration.

The *Acute Angle View* shown in *Figure 52* is established by flipping *Figure 51* horizontally to its left, and thereafter rotating it ninety degree in the clockwise direction.

Members $\overline{O'B'}$, $\overline{AB}$, and $\overline{A'B'}$ appearing in *Figure 51* are omitted. Instead linkage $\overline{O'A'}$ is extended, and member $\overline{OH}$ along with a perpendicular bisector linkage to member $\overline{O'O}$ are incorporated.

Figure 52. Atacins Car Jack Configuration.



During device flexure, triangle GOO′ resembles a *car jack configuration* as point G becomes raised or lowered relative to its base $\overline{OO'}$.

Moreover, since:

$\angle BOO' = \angle A'O'O = 2\theta$ (Ref. lines 14 and 38 of Sec. 22.6.2 Proof)

And, $\angle BOO' = \angle GOO'$ *(See Figure 52)*

$\angle A'O'O = \angle GO'O$ *(See Figure 52)*

By substitution above:

$$\angle GOO' = \angle GO'O = 2\theta$$

Therefore, triangle GOO′, as depicted in *Figure 52*, also must remain isosceles at all times.

Accordingly, point G must always reside along the perpendicular bisector to the base of isosceles triangle GOO′, designated as linkage $\overline{OO'}$.

This is achieved by introducing a slot along the longitudinal centerline of this perpendicular bisector for purposes of confining the translation of a rivet whose location always denotes point G.

In order to allow point G to be raised and lowered properly with respect to member $\overline{OO'}$, additional slots also must be embedded along the longitudinal axes of linkages $\overline{OB}$ and $\overline{O'A'}$.

The *grip* of the rivet located at point G is sized with sufficient clearance to allow all three slots which it constrains to move relative to one another in a practically frictionless manner.

One last additional slot located at point H accommodates differential lengths of member $\overline{O'H}$ that are to be incurred during flexure of the device.

This established *Car Jack Configuration* describes various angles measured from the x-axis, as notated about the circumference of a circle of radius $\overline{OA}$ which is drawn about center point O. The zero through ninety degree listings can be determined from *Figure 51*. The 180-3θ degrees angle is determined via the following proof (Ref. *Figure 52*):

ACTION	PROOF
1. $\angle O'OD = \theta$	1. Ref. line 3 of Section 22.6.2
2. $\angle BOO' = 2\theta$	2. Ref. line 14 of Section 22.6.2
3. $\angle A'O'O = 2\theta$	3. Ref. line 38 of Section 22.6.2
4. $\angle BOD = \angle BOO' + \angle O'OD$	4. The whole is equal to the sum of its parts
5. $\angle BOD = 2\theta + \theta = 3\theta$	5. Substitution of lines 1 and 2 into line 4 and summing results
6. Extend straight member $\overline{O'A'}$	6. Construction
7. Identify *point H* as the intersection between extended member $\overline{O'A'}$ and circumference of circle with center point O and radius $\overline{OA}$	7. Designation
8. Draw radius $\overline{OH}$	8. Construction
9. $\overline{OH} = \overline{OO'}$	9. All radii of a described circle are of equal length
10. Triangle OO'H is isosceles	10. An isosceles triangle contains two sides of equal length
11. $\angle OHO' = \angle HO'O$	11. Angles opposite equal sides of an isosceles triangle are equal
12. $\angle HO'O = \angle A'O'O$	12. Same angle
13. $\angle HO'O + \angle O'OH + \angle OHO' = 180^o$	13. The sum of the internal angles of triangle OO'H equal 180^o
14. $\angle HO'O + \angle O'OH + \angle HO'O = 180^o$	14. Substitution of line 11 into line 13
15. $\angle A'O'O + \angle O'OH + \angle A'O'O = 180^o$	15. Substitution of line 12 into line 14
16. $2\theta + \angle O'OH + 2\theta = 180^o$	16. Substitution of line 3 into line 15
17. $\angle O'OH = 180^o - 4\theta$	17. Subtraction of 4θ from both sides of equation
18. $\angle HOD = \angle O'OH + \angle O'OD$	18. The whole is equal to the sum of its parts
19. $\angle HOD = 180^o - 4\theta + \theta = (180 - 3\theta)^o$	19. Substitution of lines 17 and 1 into line 18; summing terms

Car Jack Configuration linkages, fully equipped with slots, may be manufactured from three-eighths inch wide by 1/24 inch thick clear polycarbonate strips.

Linkage widths may be reduced even further, to ¼ inch, by means of replacing slots by *slides* at suitable locations. Such slides operate around effected linkages allowing the later to move within them during flexure.

The *Car Jack Configuration* enables trisection to be performed for <u>both</u> *given acute* angles, as well as *given obtuse angles*. To easily track such trisections, respective linkages should be labeled as follows:

LINKAGE	**LABEL**
$\overline{OO'}$	*TRISECTOR*
$\overline{OB}$	*GIVEN ACUTE ANGLE*
$\overline{OH}$	*GIVEN OBTUSE ANGLE*

Obtuse angle trisection becomes enabled by rotating the geometry shown in the *Acute Angle View* of *Figure 52* an additional $90-3\theta$ degrees in the counterclockwise direction (Ref. *Obtuse Angle View*). Respective points upon the circle then become relocated as follows:

POINTS ALONG CIRCUMFERENCE OF CIRCLE	***Figure 52 VIEW***	
	GIVEN ACUTE ANGLE (DEGREES)	**GIVEN OBTUSE ANGLE (DEGREES)**
D	0	$0+(90-3\theta)= 90-3\theta$
O′	θ	$\theta+(90-3\theta)=90-2\theta$
B	3θ	$3\theta+(90-3\theta)=90$
A	90	$90+(90-3\theta)=180-3\theta$
H	$180-3\theta$	$180-3\theta+(90-3\theta)=270-6\theta$

With respect to *Figure 52*:

- In the *Acute Angle View*, notice that the angle described at point O′ (when measured away from the x-axis) is the ***trisector*** of the *Given Acute Angle* described at point B, such that $\theta/3\theta = 1/3$
- In the *Obtuse Angle View*, notice that the angle described at point O′ (when measured away from the x-axis) is the ***trisector*** of the *Given Obtuse Angle* described at point H, such that $(90-2\theta)/(270-6\theta) = 1/3$

The *Atacins Car Jack Configuration* introduces a *removable pin* in order to achieve trisection of both *given acute* and *given obtuse angles*. Such pin may be placed somewhere along the circumference of a circle of radius $\overline{OA}$ whose center is located at point O (Ref. *Figure 52*).

Figure 53 and *Figure 54* reflect configurations where such removable pin is located at ninety degrees. When inserted, it fixes either point A or point B of the device (see *Figure 52*) at that location. During flexure, all other linkages articulate relative to such fixed location(s).

In particular:

a) *Figure 53* portrays a ***GIVEN ACUTE ANGLE*** of 30°; whereby the ***TRISECTOR*** linkage reads 10°. The ***GIVEN OBTUSE ANGLE*** linkage thereby resides at $180-3\theta$; (i.e., 180° -30°, or 150°); and

b) *Figure 54* portrays a ***GIVEN OBTUSE ANGLE*** of 210°; whereby the ***TRISECTOR*** linkage reads 70°.

Figure 53. Atacins Car Jack Configuration Trisection of 30 Degree Acute Angle.

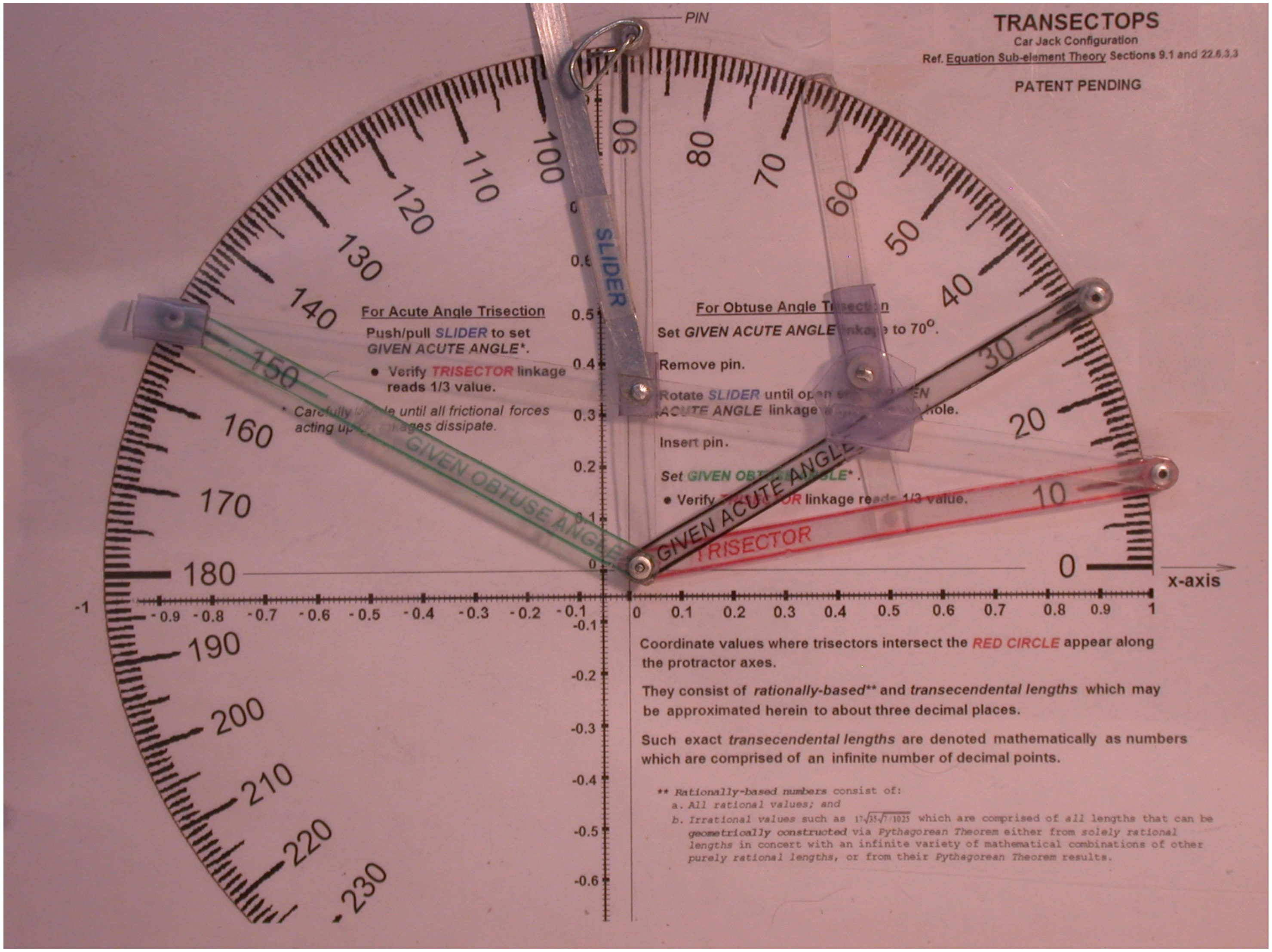
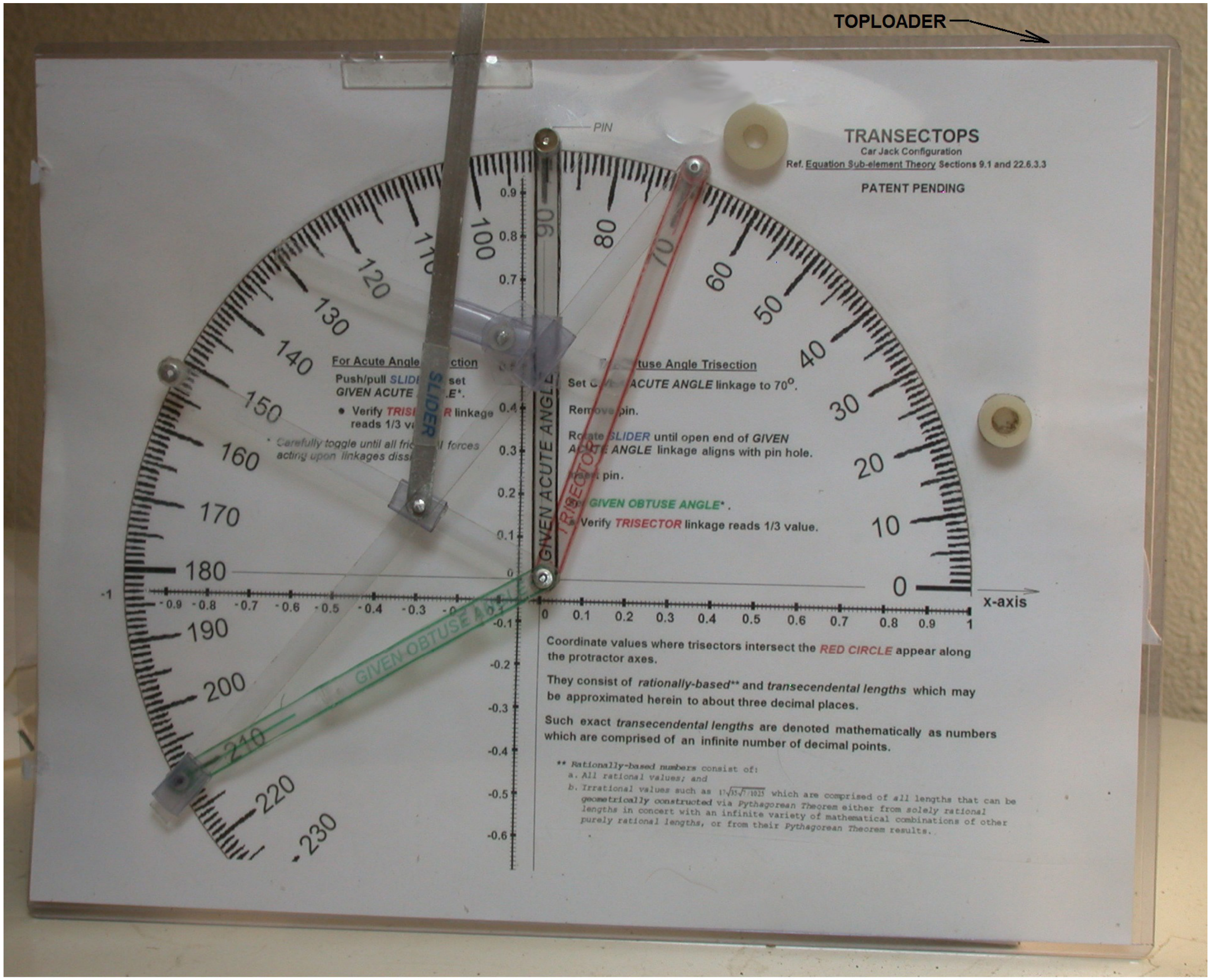


Figure 54. Atacins Car Jack Configuration Trisection of 210 Degree Obtuse Angle.



By maintaining the aggregate of all of device linkage and rivet head thicknesses to an absolute minimum, the entire *car jack configuration* may be placed into a sealed 8-1/2" by 11" *toploader*. Only the end portion of an additional *SLIDER* linkage (denoted as SLIDER L. below) permeates the toploader for purposes of setting any *given acute* or *given obtuse angle* from outside of it.

For this device, the following *stacking chart* is afforded where the bottom of the device (looking down from the top) is denoted as level 0 (Ref. *Figure 52*):

LEVEL	**LOCATION**						
	POINT O	**POINT O'**	**POINT B**	**POINT A**	**POINT H**	**POINT A'**	**POINT G**
Level 0	WASHER	WASHER	WASHER	WASHER	WASHER	N/A	N/A
Level 1	$\overline{OA}$	WASHER	WASHER	$\overline{OA}$	WASHER	$\overline{OA}$	N/A
Level 2	$\overline{OO'}$ *	$\overline{OO'}$	WASHER	N/A	WASHER	SLIDER	$\overline{OO'}$ *
Level 3	$\overline{OB}$	WASHER	$\overline{OB}$	N/A	$\overline{OH}$ **	SLIDER L.	$\overline{OB}$
Level 4	$\overline{OH}$ **	$\overline{O'H}$	N/A	N/A	$\overline{O'H}$	$\overline{O'H}$	$\overline{O'H}$

* Includes perpendicular bisector member
** Steps down from level 4 at point O to level 3 at point H

3/32" diameter blind aluminum rivets exhibiting a *grip* length of ¼" pose a suitable size and material for attaching linkages. During assembly, such grip lengths may be set according to location. They should be sized to firmly attach joints, while still permitting a practically frictionless rotation between respective linkages.

In the above chart, 3 mm ID by 7 mm OD x 0.9 mm thick nylon washers may be used to fill voids at locations where no respective linkages exist.

In *Figure 53* and *Figure 54*, notice that *sliders* located at points H, G, and A', respectively, replace slots. These may be fabricated from the same 0.2 mm thick, high impact, rigid PVC material which comprises toploaders.

Sliders reflect the following design features:

- *At point H (Ref. Figure 55)*, a 3/32" diameter circular hole is cut through a *slider (Ref. View A-A lower portion)*. Once aligned, the *grip* of a *blind rivet* is simply passed through this hole, stitched through another hole made at an end of member $\overline{OH}$, and thereafter passed through three nylon washers.

During blind rivet pull-up operations, its grip length is reduced to a degree which does not interfere with the uninhibited rotation between the captured slider and member $\overline{OH}$, but nevertheless firmly secures the joint as it becomes clamped together.

The two open ends of the slider then are folded around member $\overline{O'H}$ in opposite directions, and thereafter secured to one another by glue *(Ref. Detail A-A upper view)*. This serves to restrict translation of member $\overline{O'H}$ to the slider's longitudinal axis.

The bottom washer is countersunk on its lower side. This enables its outer edge to work in conjunction with the blind rivet bottom which is filed, sanded and then buffed to a smooth finish in order to avail a practically frictionless surface.

In View A-A of *Figure 55*, the various levels of this joint are denoted in consecutive order.

- At point G (Ref. *Figure 56)*, three sliders are riveted together. Again, the grip length is sized to allow a firm attachment and uninhibited rotation between all three. One slider is constructed in the same fashion as the slider applied at point H and rides outside the perpendicular bisector to linkage $\overline{OO'}$. The other two are mirror images of each other. This enables the flat sides, rather than the rolled sides which house the linkages, to face one another, thereby further reducing friction during device flexure.

  The various levels for this joint are afforded in the adjoining side view.

  Figure 57 presents the relative placement of point G with respect to member $\overline{OO'}$. It demonstrates why the portrayed configuration is of the same geometry specified in the above proof.

  Therein, it is illustrated that the respective longitudinal centerlines of mirror image sliders reside an arbitrary distance "r" away from rivet center located at point G'. For design purposes, an "r" of one-half inch represents a suitable length.

Length $\overline{OG'} = \overline{O'G'}$ because the rivet at point G′ is constrained to a path along the perpendicular bisector to member $\overline{OO'}$, thereby describing an isosceles triangle G′OO′ which contains two sides of equal length.

During device flexure, relative movement of linkages $\overline{OG}$ and $\overline{O'G}$ is maintained along respective longitudinal centerlines of the mirror image sliders. Their perpendicular offsets to point G′, denoted as straight line lengths $\overline{G'F}$ and $\overline{G'F'}$, respectively are equal to "r" by design, such that:

$$\sin\omega = \frac{\overline{G'F}}{\overline{OG'}} = \frac{\overline{G'F'}}{\overline{O'G'}} = \frac{r}{\overline{OG'}} = \frac{r}{\overline{O'G'}} = \sin\angle G'OG = \sin\angle G'O'G$$

Within isosceles triangle G′OO′:

$$\angle G'O'O = \angle G'OO'$$

$$\angle G'O'G + \angle GO'O = \angle G'OG + \angle GOO'$$

$$\angle G'O'G + \angle GO'O = \angle G'OG + \angle BOO'$$ (Ref. *Figure 52*)

$$\omega + \angle GO'O = \omega + 2\theta$$ (Ref. Line 14 of *Section 22.6.2 Proof*)

$$\angle GO'O = 2\theta$$

Staggering the three sliders provides a lower overall profile, whereby the rivet located at point G now occupies level 2 thru level 4 (Ref. *Figure 56* Side View). This arrangement further reduces the prospect of interference by the toploader during device flexure.

- At point A′ (Ref. *Figure 58*), the resident *slider* is located directly beneath the rivet, and in a sense presents an inverted application of the slider positioned at point H. The rivet allows an uninhibited rotation between this slider, the mid portion of linkage $\overline{O'H}$, and the end portion of a SLIDER linkage which permits manipulation of the device from outside of the toploader.

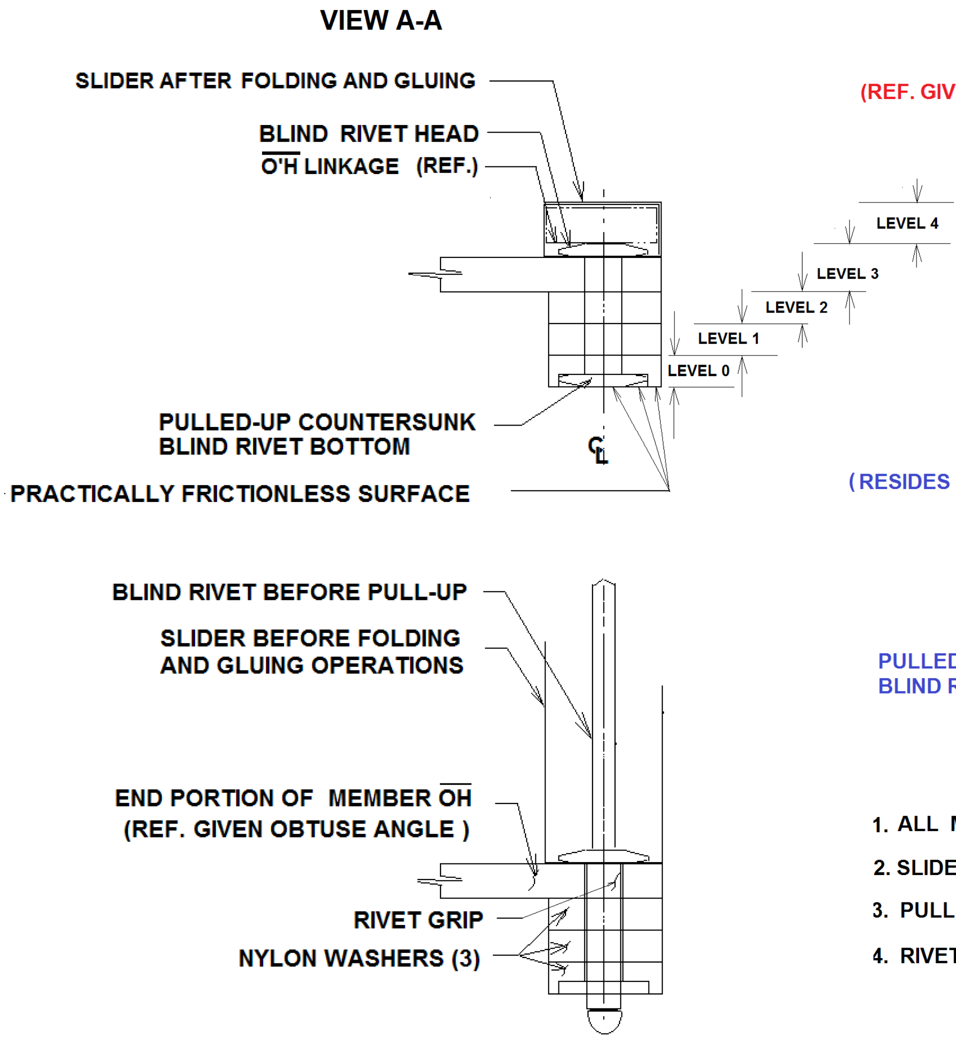
  Such rivet is inserted into a hole cut into the slider, and then threads through two 3/32″ holes – the first located a length equal to that of member $\overline{OO'}$ away from point O′ on linkage $\overline{O'H}$, and the second located at the SLIDER linkage end, respectively (Ref. *Figure 52, Figure 53*, and *Figure 54*).

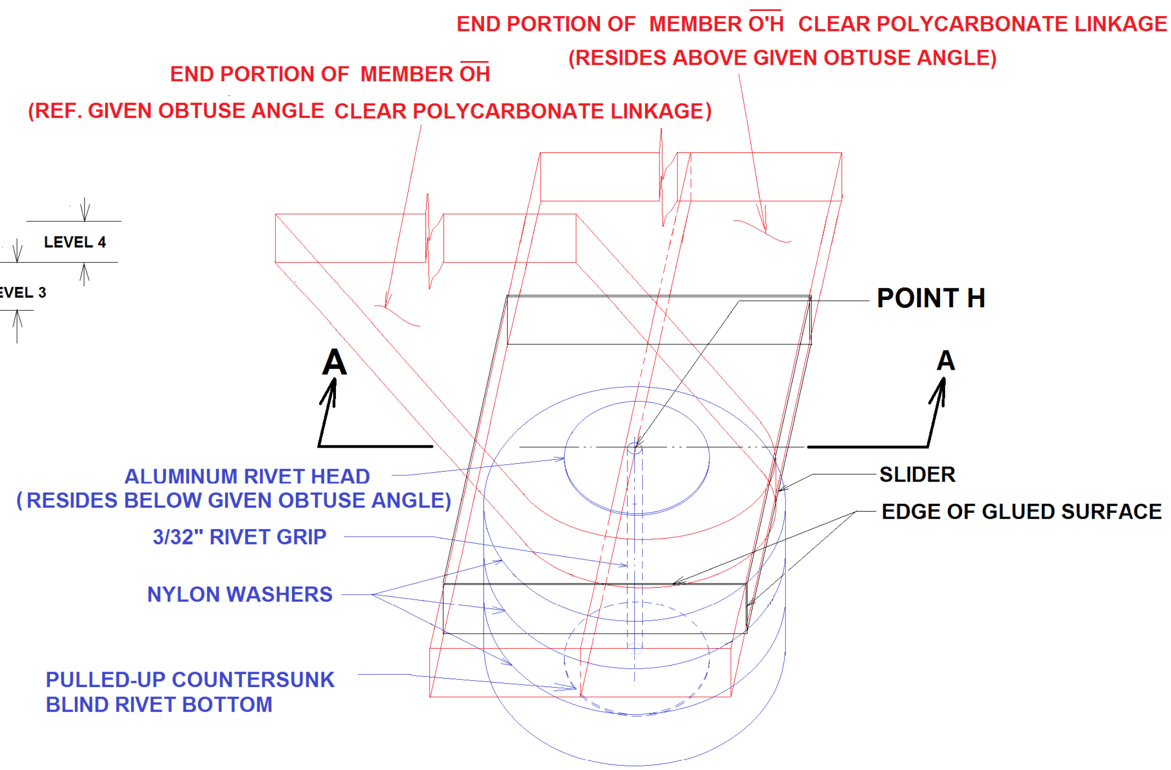
As indicated (Ref. *Figure 58*, level 0 Space), this joint does not come into contact with the bottom of the toploader, thereby enabling a more frictionless operation during device flexure.

Again, this slider is formed and glued into place after the blind rivet becomes inserted and pulled-down. Extreme caution should be exercised during gluing to make sure no glue bonds the slider to the enclosed linkage $\overline{OA}$. Also adequate space should be afforded during this procedure to assure linkage $\overline{OA}$ can move freely through the slider.

Operating instructions for the device are printed onto a sheet of paper which is inserted just above the lower sleeve of the toploader (Ref. *Figure 59*).

Figure 55. Atacins Car Jack Configuration Point H Design.





NOTES:

1. ALL MATERIALS EXCEPT 3 WASHERS AND RIVET ARE SEE-THRU.
2. SLIDER IS GLUED ALONG UPPER EXTREMITY AFTER BLIND RIVET IS PULLED-UP.
3. PULLED-UP BLIND RIVET BOTTOM IS FILED, SANDED AND THEN BUFFED TO A SMOOTH FINISH.
4. RIVET GRIP CONSTRAINS BOTTOM OF SLIDER, GIVEN OBTUSE ANGLE LINKAGE AND THREE WASHERS

Figure 56. Atacins Car Jack Configuration Point G View.

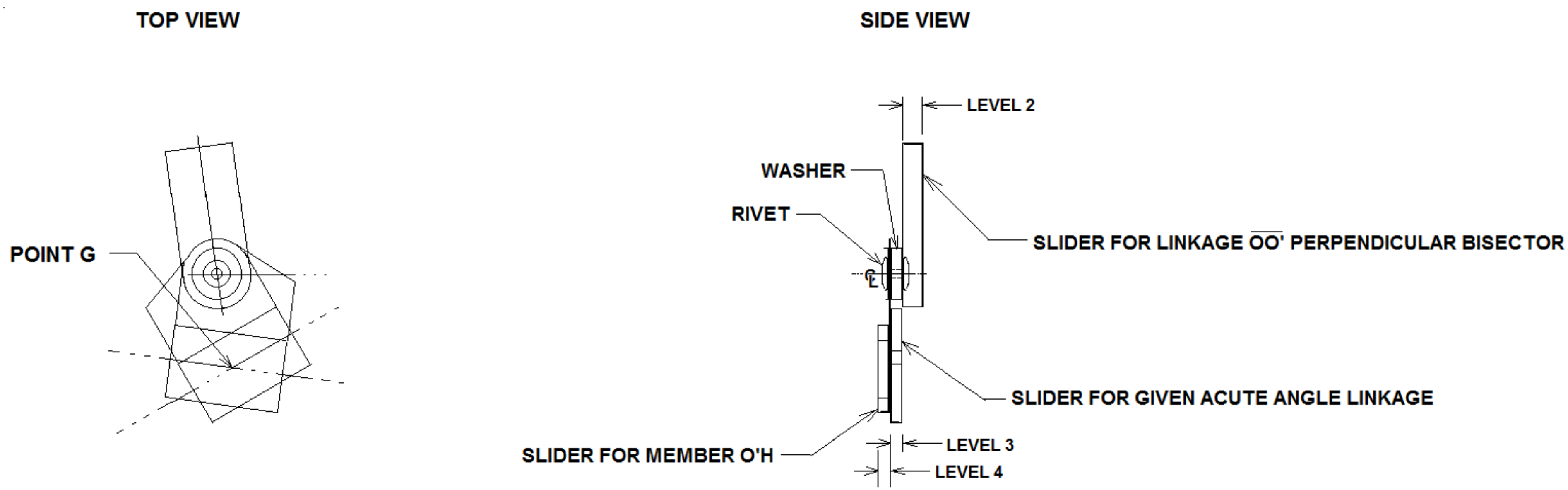


Figure 57. Atacins Car Jack Configuration Point G Relative Placement.

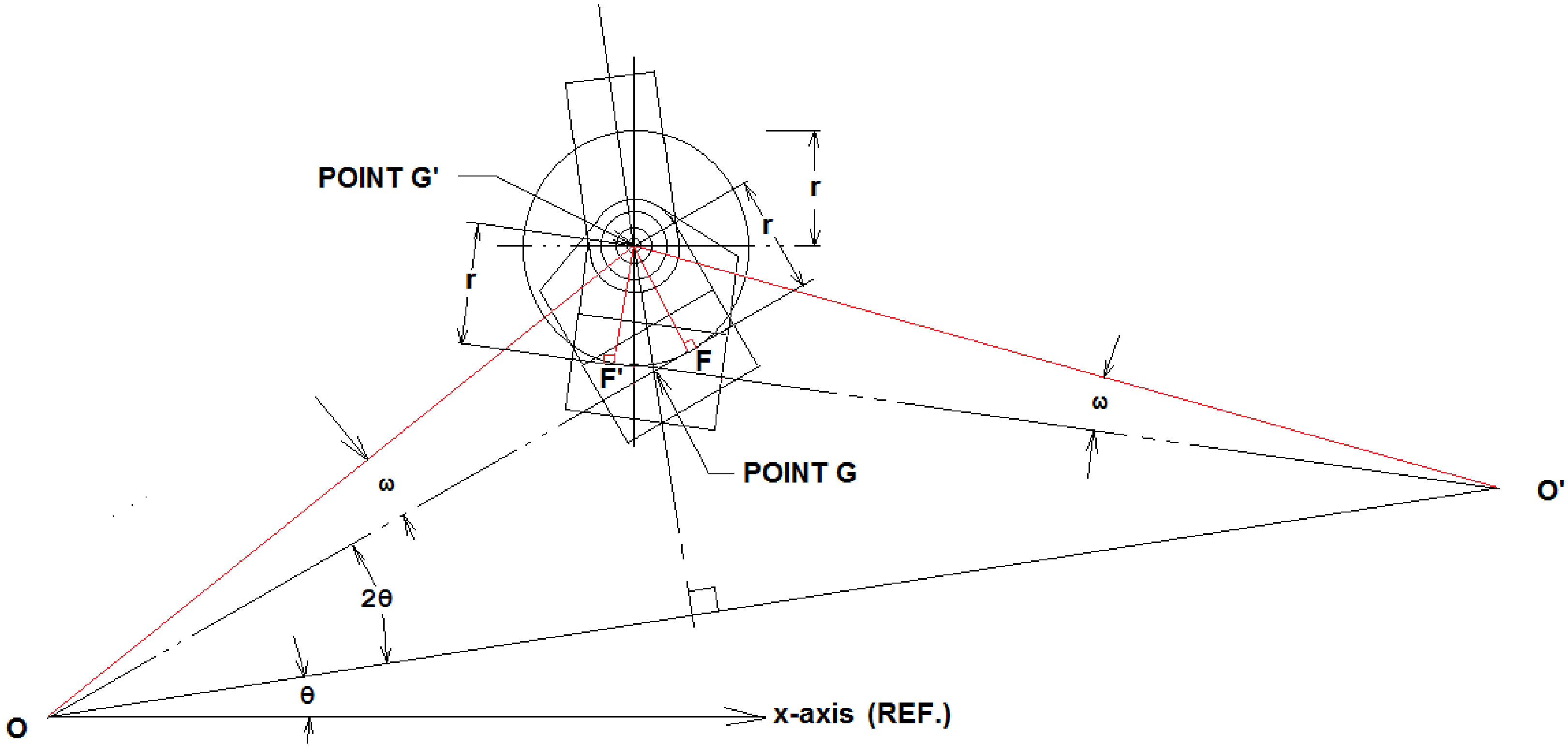


Figure 58. Atacins Car Jack Configuration Point A' Front View.

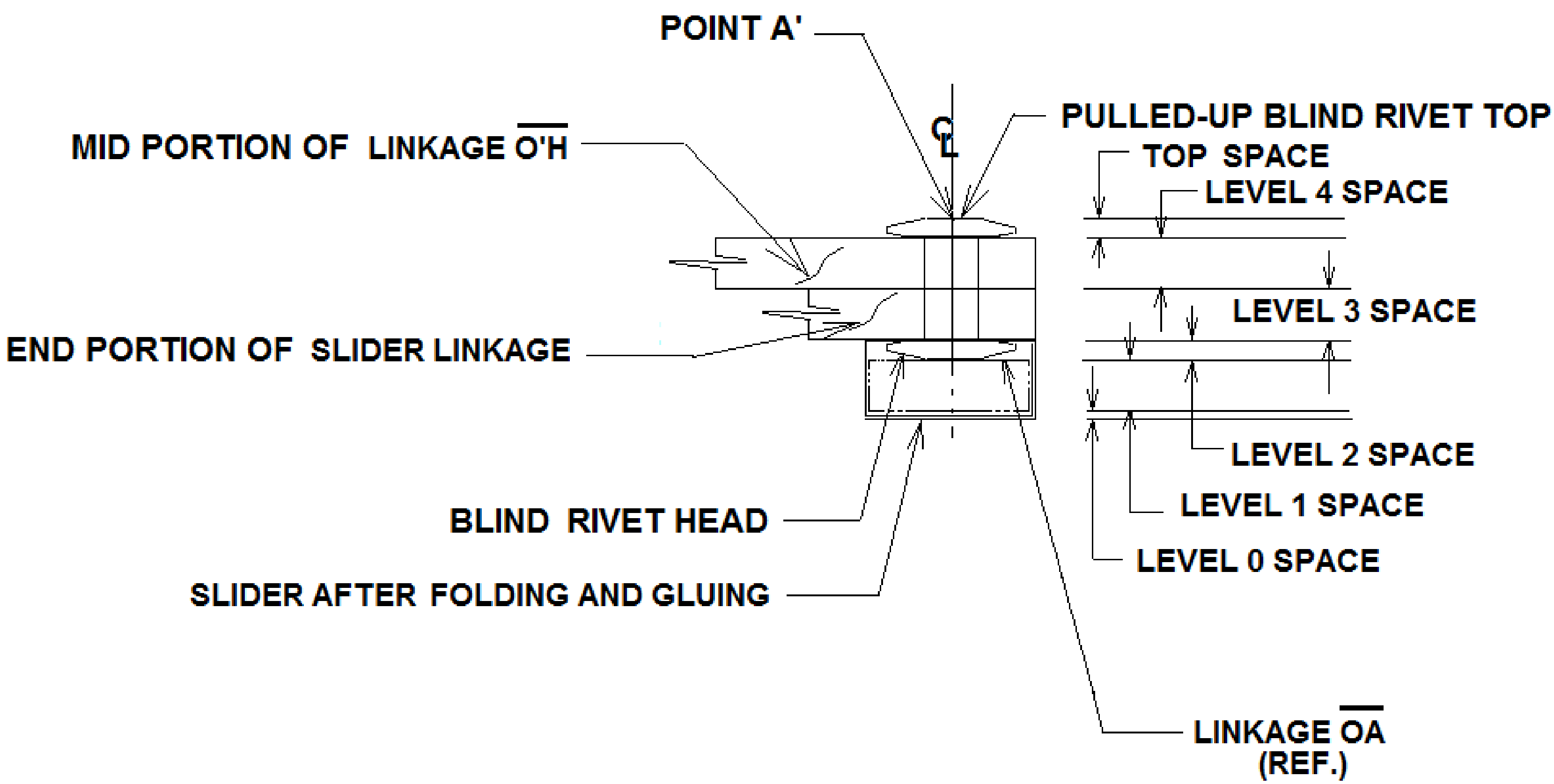
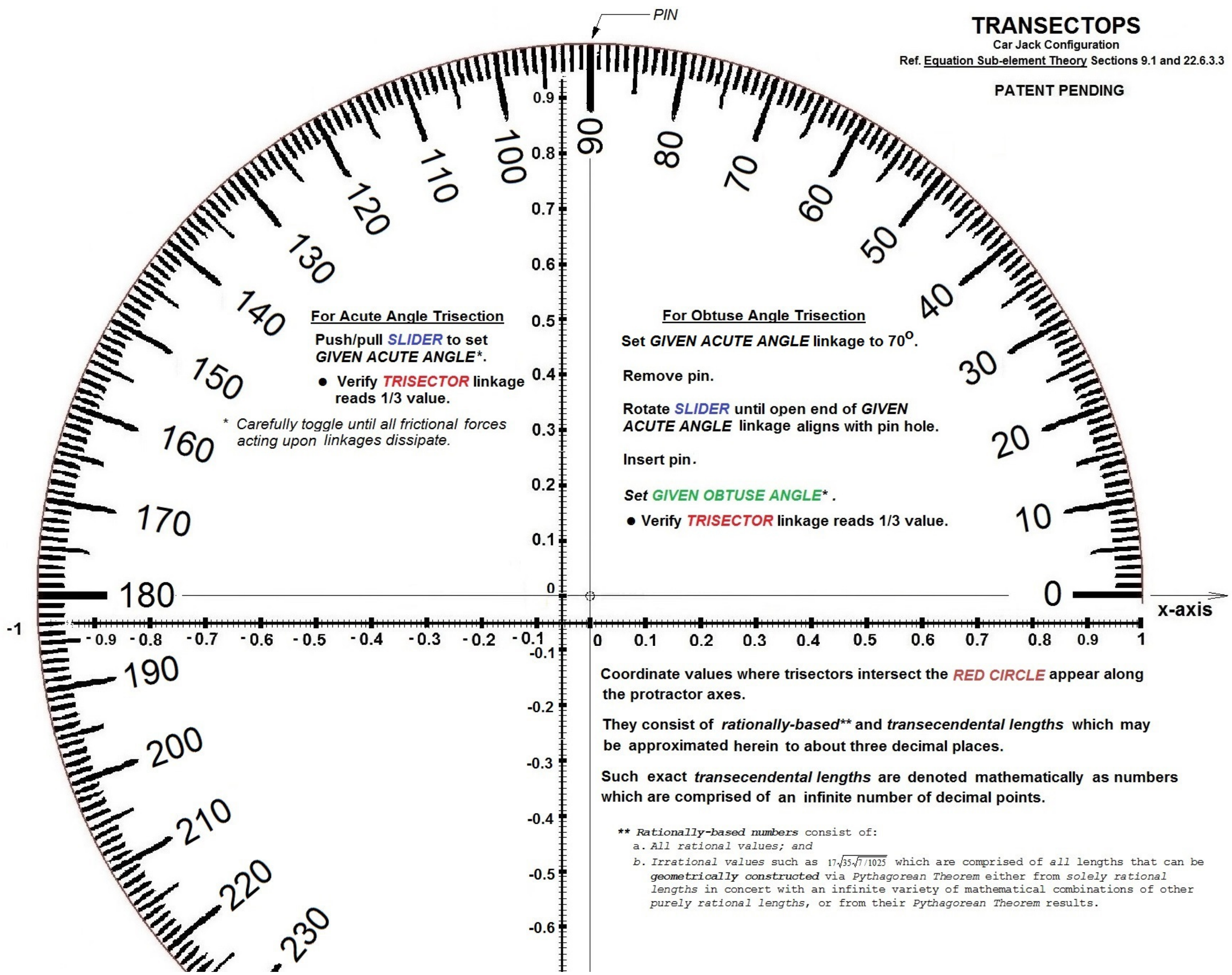


Figure 59. Atacins Car Jack Configuration Operating Instructions.



22.6.4. Exact Depictions versus Approximations.

Atacins enables *cubic irrational lengths* to be depicted as ***exact measurements***.

Since *cubic irrational lengths* describe *decimal sequences* which are considered to *continue on indefinitely*, instead of repeating themselves, oftentimes approximation techniques, like the one described below, have been administered to replicate them:

> *Dividing up a given length of unity into ten equal portions (Ref. Figure 45), and then into hundredths (Ref. Figure 46), and so on, until such desired cubic irrational length becomes amply gauged via ruler.*

Exact ***geometrically formed*** depictions do a much better job than such *approximation* techniques. Hence, in many cases, mathematicians should consider the latter as being *obsolete.*

22.7. Summary.

To summarize, the *Atacins* features only *compass* and *straightedge* construction where actuation proceeds from completely ***identifiable locations***.

In retrospect, such *Atacins approach* is far different from the sliding *straightedge* operation devised by *Archimedes*, where alignment follows no *prescribed geometric pattern*, but instead is achieved only after engaging upon a process of *trial and error*.

Such uncontrolled movement rendered by *Archimedes* is easily validated in the *simplified construction* stipulated below which *disposes* of the need to construct the *parallel straight line* alluded to in the *aforementioned Neusis construction (Ref. Figure 51)*:

> A *straightedge* is to operate in concert with *isosceles triangle* AOB whose $\angle AOB$ is *given* as $(90-3\theta)^o$. The *straightedge* is *notched* somewhere along it in order to mark off, or designate a *length* $\overline{EF}=\overline{OA}=\overline{OB}$. While passing directly over Point B such *straightedge* is to be *aligned* until such time that its Point E superimposes itself somewhere along the circumference of a circle of radius $\overline{OA}$ as described about Point O, while its Point F resides upon straight line $\overline{OA}$ extended.

To reiterate, *Atacins* actuations are launched exclusively from known locations. No *guess work* is required!

Based upon the unique manner in which *Atacins* obtains ***geometrically formed*** angles, *rationally-based lengths* can be depicted directly alongside associated *cubic irrational root lengths.*

There is no need to change the wording of the *conclusion* to *Section 9.1* since it remains consistent with the above logic because:

- ***Rationally-based numbers*** comprise all *real numbers* which can be ***geometrically constructed*** from a given, arbitrary length of unity
- ***Cubic irrational numbers*** comprise all other *real numbers*; specifically, those which cannot be ***geometrically constructed*** from a given, arbitrary length of unity – which includes all those which can be ***geometrically formed*** from a given, arbitrary length of unity

Lastly, *Atacins* can be used to obtain a ***geometrically formed*** trisector for *any given angle* simply by applying the following two step process *(Ref. Figure 51)*:

1) Set *angles AOB and A'O'B' to predetermined* angles of *90–3θ degrees* each;
2) Then articulate, or flex the invention until such time that the *longitudinal axis* of *member* $\overline{O'B'}$ ***overlaps*** *point B*.

The trisected angle OO'C thereafter becomes easily identified by *bisecting* the ***geometrically formed*** *angle OO'A'*.

This can be accomplished either by the use of added pencil and paper, or by aligning a separate straightedge with point O' and rotating it until it becomes perpendicular to *member* $\overline{OA}$.

Three other alternative designs also are presented. In the last, all that is needed is to set any *given acute*, or *given obtuse angle* (Ref. *Figure 52, Figure 53* and *Figure 54*). The *TRISECTOR* linkage then automatically articulates to an angle which is exactly one-third its magnitude.

SECTION 23. *MOVING WAVES.*

Moving waves may be modeled by making use of *associated RST curves.* Frictional effects which act upon such waves over time have the ability to change their shape during their dissipation.

A *shock wave* presents itself as a holograph of forces. As it moves through *different molecular mediums*, it in turn is further acted upon during its *normal expansion* process. Any given wave pattern has its own set of individual characteristics which causes it to ultimately disintegrate here on earth in a unique fashion. Some of these characteristics may be variations in pressure, temperature, and density, just to name a few.

Charting these waves accurately may pose a significant challenge. However, it is contended that some of the tools established in this treatise should enable such a proposed mapping.

The particular example presented directly below is easy for *mathematicians* to comprehend and identify with. This is because it charts *human people*, rather than molecules.

It consists of a group of people termed *offensive linemen* who are playing the *game of football.* The term offensive applies to the fact that they are in possession of the ball, not that they are fowl smelling.

As play commences, they move in particular fashion. Moreover, patterns associated with protecting a *quarterback* who just so happens to possess the *football* pertain to situations when he is *safely surrounded,* or *protected* in a so-called *pocket.* Such pocket invariably may be likened to the *eye of a hurricane* which occupies a rather peaceful existence; while outside of it, the world is subject to incredible turbulence.

For charting purposes, this example relates curves whose respective origins all are located directly in front of the quarterback's starting point, at the very near of the center of the pocket, in order to *emphasize symmetry*.

However, depending upon the play being performed and the intended location where the *quarterback* ultimately plans to finally reside just before throwing the football, this perspective may change.

An animation thereof may be viewed by *clicking* on the website link entitled, "Truescans/FB.htm" in order that it should play on another screen in your *browser*.

The *line of scrimmage* signifies an imaginary line which separates opposing teams before each play commences. It passes directly in front of where the *center offensive lineman* is located, or is *lined up*, before play begins.

A *Straight Drop-back* scenario is afforded, which means that the *quarterback* moves *directly backwards* from where he is initially positioned with respect to the *line of scrimmage* after play commences. Play begins with the *snap* of the football at which time such respective *movement* between players takes place.

For this study, the *relative movement* of offensive players also remains perpendicular to the x-axis or *line of scrimmage*. Hence, they all remain in, or *cover* certain *slots* while dropping back behind the *line of scrimmage*.

A *singular curve* exists which defines the locations where all linemen reside at any given instance in time.

Before play, such curve is skewed slightly behind the *line of scrimmage* in order to discourage linemen from lining up *off sides*, which would result in a penalty.

After play commences, *respective curves* become steeper in order to allow the offensive linemen to remain in the direct path between the quarterback and opposing players.

Respective curves can be adjusted during the programming sequence to account for how fast the *quarterback* is simultaneously retreating.

The depth away from *the line of scrimmage* that the *quarterback* wishes to traverse represents another consideration, or variable, which can be programmed into the play arrangement.

This proposed, ***overall football program*** enables strategies, or plays, to be assessed with respect to the amount of time needed for players to move to specified positions.

The degree of separation maintained between offensive linemen can be scrutinized in detail in order to determine optimal spacings for running back penetration, while limiting hole sizes to those which best discourage opposing player advancements.

New ***sub-element applications*** for moving waves focus upon the fact that they permit an *intrusion* of unique thermodynamic properties to invade a fluid or gaseous medium such as water or air. Whether or not such medium proves to be incompressible has impact upon the final physical outcome.

As a moving wave becomes propelled into this medium, it is acted upon by resident pressures and temperatures which may be assessed in terms of unique characteristics within the medium such as flow rates, densities, wind velocities, etc.

In a pure vacuum, an intruding wave that reaches equilibrium may permeate in an undisturbed fashion throughout time, ultimately traveling to the outer reaches of an infinite space. In such cases, it could be viewed as a 'standing wave'; i.e., one which retains a *fixed geometry* throughout its travel.

Otherwise, the wave becomes *diminished* by the forces acting upon it or enlarged by forces acting within it. It changes shape and may be characterized as a 'moving wave' which ultimately assumes the form of various waves, or curve patterns throughout its life. As a curve front changes shape, a history of the forces which have acted upon it may be traced to the change in curve pattern observed!

The moving wave portrayed at time t = 0 in *Figure* changes shape as it travels through a medium. For this particular wave, all intermediate portions move relative to points A and B as the wave disintegrates from time t= 0 to time t = t_2. In order to do so, during circumstances when point A moves, point B would have to move in much the same way in order to maintain the same distance away from it. Node O, shown to be located in the middle of the symmetrical wave, travels through point O' at time t = t_1 to a location of O'' at time t = t_2. As such, the wave appears to be collapsing, as opposed to expanding. This presents an indication either that:

- Weaker resisting wave forces are at play at the vertical plane that node O passes through as it moves from Node O′ to Node O′′ than those tending to resist points A and B from propagating during this same time period,

- Weaker applied *thermodynamic forces* reside at endpoints A and B of the moving wave than at its apex, or
- Any combination thereof

The term *plane* mentioned above applies to the fact that moving waves, such as that represented in *Figure* , assume three dimensional shapes in the real world. Their respective cross sections may be circular, elliptical, or any other variation that conceptually may be modeled over time. Furthermore, such cross sections may change shape affording additional provision in which to characterize the forces at work.

As such, <u>it is contended</u> that such process may be employed to help validate just how and why moving waves *whose changing shapes may be described by a series of charted curves* are *thermodynamically impacted* when traveling through various mediums.

Figure 60. Wave Moving through a Medium Portrayed at times t , t_1, and t_2.